MODERN ALGEBRA

WELLS' MATHEMATICAL SERIES.

A

COMPLETE COURSE

IN

ALGEBRA

FOR

ACADEMIES AND HIGH SCHOOLS.

BY

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PREFACE.

THE present work contains a full and complete treatment of the topics usually included in an Elementary Algebra. The author has endeavored to prepare a course sufficiently advanced for the best High Schools and Academies, and at the same time adapted to the requirements of those who are preparing for admission to college.

Particular attention has been given to the selection of examples and problems, a sufficient number of which have been given to afford ample practice in the ordinary processes of Algebra, especially in such as are most likely to be met with in the higher branches of mathematics. Problems of a character too difficult for the average student have been purposely excluded, and great care has been taken to obtain accuracy in the answers.

The author acknowledges his obligations to the elementary text-books of Todhunter and Hamblin Smith, from which much material and many of the examples and problems have been derived. He also desires to express his thanks for the assistance which he has received from experienced teachers, in the way of suggestions of practical value.

WEBSTER WELLS.

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ALGEBRA.

I. DEFINITIONS AND NOTATION.

1. **Algebra** is that branch of mathematics in which the relations of numbers are investigated, and the reasoning abridged and generalized by means of *symbols*.

Note. Writers on Algebra employ the word "quantity" as synonymous with "number"; this definition of the word will be understood throughout the present work.

2. The **Symbols** of Algebra are of four kinds:

1. Symbols of Quantity.
2. Symbols of Operation.
3. Symbols of Relation.
4. Symbols of Abbreviation.

SYMBOLS OF QUANTITY.

3. The symbols of quantity generally used are the *figures* of Arithmetic, and the *letters* of the alphabet.

Figures are used to represent known quantities and determined values; while letters may represent any quantities whatever, known or unknown.

4. **Known Quantities**, or those whose values are given, when not expressed by figures, are usually represented by the first letters of the alphabet, as a, b, c.

5. **Unknown Quantities**, or those whose values are to be determined, are usually represented by the last letters of the alphabet, as x, y, z.

6. Quantities occupying similar relations in the same problem, are often represented by the same letter, distinguished by different *accents;* as a', a'', a''', read "a prime," "a second," "a third," etc.

They may also be distinguished by different *subscript* figures; as a_1, a_2, a_3, read "a one," "a two," "a three," etc.

7. Zero, or the absence of quantity, is represented by the symbol 0.

SYMBOLS OF OPERATION.

8. The **Sign of Addition,** $+$, is called "*plus.*"

Thus, $a+b$, read "a plus b," indicates that the quantity b is to be added to the quantity a.

9. The **Sign of Subtraction,** $-$, is called "*minus.*"

Thus, $a-b$, read "a minus b," indicates that the quantity b is to be subtracted from the quantity a.

Note. The sign $\sim$ indicates the difference of two quantities; thus, $a \sim b$ denotes that the difference of the quantities a and b is to be found.

10. The **Sign of Multiplication,** $\times$, is read "*times,*" "*into,*" or "*multiplied by.*"

Thus, $a \times b$ indicates that the quantity a is to be multiplied by the quantity b.

The sign of multiplication is usually omitted in Algebra, except between arithmetical figures; the multiplication of quantities is therefore indicated by the absence of any sign between them. Thus, $2ab$ indicates the same as $2 \times a \times b$.

A point is sometimes used in place of the sign $\times$ between two or more figures; thus, $2 \cdot 3 \cdot 4$ denotes $2 \times 3 \times 4$.

11. Quantities multiplied together are called *factors,* and the result of the multiplication is called the *product.*

Thus, 2, a, and b are the factors of the product $2ab$.

12. A **Coefficient** is a number prefixed to a quantity to indicate how many times the quantity is to be taken.

Thus, in $4ax$, 4 is the coefficient of ax, and indicates that ax is to be taken 4 times; that is, $4ax$ is equivalent to $ax + ax + ax + ax$.

When no coefficient is expressed, 1 is understood to be the coefficient. Thus, a is the same as $1a$.

When any number of factors are multiplied together, the product of any of them may be regarded as the coefficient of the product of the others. Thus, in $abcd$, ab is the coefficient of cd; b of acd; abd of c; etc.

13. An **Exponent** is a figure or letter written at the right of, and above a quantity, to indicate the number of times the quantity is taken as a factor.

Thus, in x^3, the ^{3} indicates that x is taken three times as a factor; that is, x^3 is equivalent to xxx.

14. The product obtained by taking a factor two or more times is called a *power*. A single letter is also often called the *first* power of that letter. Thus,

a^2 is read "a to the second power," or "a *square*," and indicates aa;

a^3 is read "a to the third power," or "a *cube*," and indicates aaa;

a^4 is read "a to the fourth power," or "a *fourth*," and indicates $aaaa$; etc.

When no exponent is written, the *first* power is understood; thus, a is the same as a^1.

15. The **Sign of Division,** $\div$, is read "*divided by.*"

Thus, $a \div b$ denotes that the quantity a is to be divided by the quantity b.

Division is also indicated by writing the dividend above, and the divisor below, a horizontal line. Thus, $\frac{a}{b}$ indicates the same as $a \div b$. When thus written, $\frac{a}{b}$ is often read "*a over b.*"

SYMBOLS OF RELATION.

16. The symbols of relation are signs used to indicate the relative magnitudes of quantities.

17. The **Sign of Equality,** $=$, read "*equals,*" or "*is equal to,*" indicates that the quantities between which it is placed are equal.

Thus, $x = y$ indicates that the quantities x and y are equal.

A statement that two quantities are equal is called an *equation.*

Thus, $x + 4 = 2x - 1$ is an equation, and is read "x plus 4 equals $2x$ minus 1."

18. The **Sign of Inequality,** $>$ or $<$, read "*is greater than*" and "*is less than*" respectively, when placed between two quantities, indicates that the quantity toward which the opening of the sign turns is the greater.

Thus, $x > y$ is read "x is greater than y"; $x - 6 < y$ is read "x minus 6 is less than y."

SYMBOLS OF ABBREVIATION.

19. The **Sign of Deduction,** $\therefore$, stands for *therefore* or *hence.*

20. The **Signs of Aggregation,** the *parenthesis* ( ), the *brackets* [ ], the *braces* { }, and the *vinculum* ——, indicate that the quantities enclosed by them are to be taken collectively. Thus,

$$(a + b)x, \quad [a + b]x, \quad \{a + b\}x, \quad \overline{a + b} \times x,$$

all indicate that the quantity obtained by adding a and b is to be multiplied by x.

21. The **Sign of Continuation,** ..., stands for "*and so on*" or "*continued by the same law.*" Thus,

$$a, \ a+b, \ a+2b, \ a+3b, \ \ldots$$

reads "a, a plus b, a plus $2b$, a plus $3b$, and so on."

ALGEBRAIC EXPRESSIONS.

22. An **Algebraic Expression** is any combination of algebraic symbols; as $2x^2 - 3ab + c^3$.

23. A **Term** is an algebraic expression whose parts are not separated by the signs $+$ or $-$; as $2x^2$, $-3ab$, or $+c^3$.

$2x^2$, $-3ab$, and c^3 are called the terms of the expression $2x^2 - 3ab + c^3$.

24. **Positive Terms** are those preceded by a *plus* sign; as

$$+2x^2, \text{ or } +c^3.$$

For this reason, the sign $+$ is often called the *positive sign.* If no sign is expressed, the term is understood to be positive; thus, a is the same as $+a$.

25. **Negative Terms** are those preceded by a *minus* sign; as

$$-3ab, \text{ or } -bc^2.$$

For this reason, the sign $-$ is often called the *negative sign;* it can never be omitted before a negative term.

Note. In a negative term, the numerical coefficient indicates how many times the quantity is to be taken *subtractively.* (Compare Art. 12.)

Thus, $-3ab$ is equivalent to $-ab-ab-ab$.

26. In Arithmetic, if the same number be both added to and subtracted from another, the value of the latter will not be changed. Thus,

$$5+3-3=5.$$

Similarly, in Algebra, if any quantity b be both added to and subtracted from another quantity a, the result will be equal to a. That is,

$$a + b - b = a.$$

Consequently, equal terms affected by *unlike signs*, in an expression, neutralize each other, or *cancel*.

27. A **Monomial** is an algebraic expression consisting of only one term; as $5a$, $7ab$, or $-3b^2c$.

A monomial is sometimes called a *simple* quantity.

28. A **Polynomial** is an algebraic expression consisting of more than one term; as $a + b$, or $3a^2 + b - 5b^3$.

A polynomial is sometimes called a *compound* quantity.

29. A **Binomial** is a polynomial of two terms; as $a - b$, or $2a + b^2$.

30. A **Trinomial** is a polynomial of three terms; as $ab + 2c^2 - b^3$.

31. **Similar** or **Like Terms** are those which differ only in their numerical coefficients. Thus,

$2xy^2$ and $-7xy^2$ are similar terms.

32. **Dissimilar** or **Unlike Terms** are those which are not similar. Thus,

bx^2y and bxy^2 are dissimilar terms.

33. The **Degree** of a term is the number of *literal* factors which it contains. Thus,

$2a$ is of the *first* degree, since it contains but *one* literal factor;

ab is of the *second* degree, since it contains *two* literal factors;

$3a^2b^3$ is of the *fifth* degree, since it contains *five* literal factors.

The degree of any term is determined by adding the exponents of its several letters. Thus,

ab^2c^3 is of the *sixth* degree.

34. Homogeneous Terms are those of the same degree. Thus, a^2, $3bc$, and $-4x^2$ are homogeneous terms.

35. A polynomial is called homogeneous when all its terms are homogeneous; as $a^3 + 2abc - 3b^3$.

36. A polynomial is said to be *arranged* according to the *ascending* powers of any letter, when the term containing the lowest exponent of that letter is placed first, that having the next higher immediately after, and so on. Thus,

$$b^4 + 3ab^3 - 2a^2b^2 + 3a^3b - 4a^4$$

is arranged according to the ascending powers of a.

Note. The term b^4, which does not involve a at all, is regarded as containing the lowest exponent of a in the above expression.

37. A polynomial is said to be arranged according to the *descending* powers of any letter, when the term containing the highest exponent of that letter is placed first, that having the next lower immediately after, and so on. Thus,

$$b^4 + 3ab^3 - 2a^2b^2 + 3a^3b - 4a^4$$

is arranged according to the descending powers of b.

38. The **Reciprocal** of a quantity is 1 divided by that quantity. Thus, the reciprocal of a is $\frac{1}{a}$; and of $x + y$ is $\frac{1}{x+y}$.

39. The **Numerical Value** of an expression is the result obtained by rendering it into an arithmetical quantity, by means of the numerical values assigned to its letters.

Thus, the numerical value of

$$4a + 3bc - d^3$$

when $a = 4$, $b = 3$, $c = 5$, and $d = 2$, is

$$4 \times 4 + 3 \times 3 \times 5 - 2^3 = 16 + 45 - 8 = 53.$$

EXERCISES.

40. Find the numerical value of the following expressions, when $a = 2$, $b = 3$, $c = 1$, and $d = 4$:

1. $a^2 + 2ab - c + d.$

2. $3a^3 - 2a^2b + c^3.$

3. $5a^2b + 4ab^2 - 27cd.$

4. $2a^2 + 3bc - \dfrac{5}{cd}.$

5. $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{d}.$

6. $\dfrac{b^a}{a^d}.$

7. $\dfrac{cd}{b^2} + \dfrac{ab}{c^2}.$

8. $b^d - a^bb^2.$

9. $\dfrac{3d^2}{5ac} - \dfrac{2a}{3b^2}.$

10. $\dfrac{a^2}{b^2} + \dfrac{b^2}{c^2} + \dfrac{c^2}{d^2}.$

If the expression involves parentheses, the operations indicated *within* the parentheses must be performed *first*.

Thus, to find the numerical value, when $a = 3$, $b = 2$, and $c = 1$, of

$$b(2a - 3c)(a^2 + c^2) - \frac{a^2 + b^2}{a^2 - b^2},$$

we have,

$$\begin{aligned} 2a - 3c &= 6 - 3 = 3 \\ a^2 + c^2 &= 9 + 1 = 10 \\ a^2 + b^2 &= 9 + 4 = 13 \\ a^2 - b^2 &= 9 - 4 = 5 \end{aligned}$$

Hence the numerical value of the expression is

$$2 \times 3 \times 10 - \frac{13}{5} = 60 - \frac{13}{5} = \frac{287}{5}.$$

Find the numerical values of the following, when $a=4$, $b=2$, $c=3$, and $d=1$:

11. $a^2(a+b)-2abc.$

12. $7a^2+(a-b)(a-c).$

13. $15a-7(b+c)-d.$

14. $c(a^{b+d}+a^{b-d}).$

15. $25a^2-7(b^2+c^2)+d^2.$

16. $\dfrac{4}{3a-3c}+\dfrac{8}{3}.$

17. $\dfrac{25a-30c-d}{b+c}.$

18. $\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{c^2-b^2}{c^2+b^2}.$

Find the values of the following, when $a=\dfrac{1}{2}$, $b=\dfrac{1}{3}$, $c=\dfrac{1}{5}$, and $x=2$:

19. $(2a+3b+5c)(8a+3b-5c)(2a-3b+15c).$

20. $x^3+\left(\dfrac{1}{a}+\dfrac{1}{b}\right)x^2+\left(\dfrac{1}{b}-\dfrac{1}{a}\right)x+\dfrac{2}{b^2}.$

21. $x^4-(2a+3b)x^3+(3a-2b)x^2-cx+bc.$

22. $\dfrac{a^2-\dfrac{b}{2}}{8bc-a}-\dfrac{x}{a+b+c}.$

41. Put the following into the form of algebraic expressions:

1. Five times a, added to twice b.

2. Two times x, minus y to the second power.

3. The product of a, b, c square, and d cube.

4. Three times the cube of a, minus twice the product of a square and b, plus the cube of c.

5. The product of $x+y$ and a.

6. The product of $x+y$ and $a-b$.

7. a square, divided by the product of b and c.

8. a square, divided by $b-c$.

9. x divided by 3, plus 2, equals three times y minus 11.

10. The product of m and $a+b$ is less than the reciprocal of x cube.

AXIOMS.

42. An **Axiom** is a truth assumed as self-evident.

Algebraic operations are based upon definitions, and the following axioms:

1. If equal quantities be added to equal quantities, the sums will be equal.
2. If equal quantities be subtracted from equal quantities, the remainders will be equal.
3. If equal quantities be multiplied by equal quantities, the products will be equal.
4. If equal quantities be divided by equal quantities, the quotients will be equal.
5. If the same quantity be both added to and subtracted from another, the value of the latter will not be changed.
6. If a quantity be both multiplied and divided by another, the value of the former will not be changed.
7. Quantities which are equal to the same quantity are equal to each other.

SOLUTION OF PROBLEMS BY ALGEBRAIC METHODS.

43. The following examples will illustrate the application of the notation of Algebra in the solution of problems.

1. The sum of two numbers is 30, and the greater is 4 times the less. What are the numbers?

We will first solve the problem by the method of Arithmetic, and afterwards by Algebra. The marginal letters refer to the corresponding steps of the two methods; that is, the operation (a) in the algebraic solution is equivalent to the operation (a) in the arithmetical; and so on. In this way the student can compare the two processes step by step.

Solution by Arithmetic.

The less number, plus the greater number, equals 30.
(*a*) Hence the less number, plus 4 times the less number, equals 30
(*b*) Therefore 5 times the less number equals 30.
(*c*) Hence the less number is one-fifth of 30, or 6.
(*d*) Then the greater number is 4 times 6, or 24.

Solution by Algebra.

Let	$x =$ the less number.
Then	$4x =$ the greater number.
(*a*) By the conditions,	$x + 4x = 30.$
(*b*) Or,	$5x = 30.$
(*c*) Dividing by 5,	$x = 6$, the less number.
(*d*) Whence,	$4x = 24$, the greater number.

2. A, B, and C together have \$66. A has one-half as much as B, and C has as much as A and B together. How much has each?

Let	$x =$ the number of dollars A has.
Then	$2x =$ the number of dollars B has,
and	$x + 2x$, or $3x =$ the number of dollars C has.
By the conditions,	$x + 2x + 3x = 66.$
Or,	$6x = 66.$
Whence,	$x = 11$, the number of dollars A has.
Therefore,	$2x = 22$, the number of dollars B has,
and	$3x = 33$, the number of dollars C has.

3. The sum of the ages of A and B is 109 years, and A is 13 years younger than B. What are their ages?

Let	$x =$ the number of years in A's age.
Then	$x + 13 =$ the number of years in B's age.
By the conditions,	$x + x + 13 = 109.$
Or,	$2x + 13 = 109.$
Whence,	$2x = 96.$
And,	$x = 48$, the number of years in A's age
Therefore,	$x + 13 = 61$, the number of years in B's age.

PROBLEMS.

4. The greater of two numbers is 5 times the less, and their sum is 42. What are the numbers?

5. The sum of the ages of A and B is 68 years, and B is 6 years older than A. What are their ages?

6. Divide $1200 between A and B, so that A may receive $128 less than B.

7. A man had $3.72; after spending a certain sum, he found that he had left 3 times as much as he had spent. How much had he spent?

8. Divide $260 between A, B, and C, so that B may receive 3 times as much as A, and C 3 times as much as B.

9. Divide the number 125 into two parts, one of which is 21 less than the other.

10. The sum of three numbers is 98; the second is 3 times the first, and the third exceeds the second by 7. What are the numbers?

11. A, B, and C together have $127; C has twice as much as A, and $13 more than B. How much has each?

12. My horse, carriage, and harness together are worth $400. The horse is worth 11 times as much as the harness, and the carriage is worth $175 less than the horse. What is the value of each?

13. The sum of three numbers is 108. The first is one-third of the second, and 33 less than the third. What are the numbers?

14. Divide the number 210 into three parts, such that the first is one-half of the second, and one-third of the third.

15. A man bought a cow, a sheep, and a hog, for $75; the price of the sheep was $27 less than the price of the cow, and $6 more than the price of the hog. What was the price of each?

NEGATIVE QUANTITIES.

44. The signs $+$ and $-$, besides indicating the operations of addition and subtraction, are also used, in Algebra, to distinguish between quantities which are the exact reverse of each other in quality or condition.

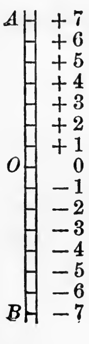
Thus, in the thermometer, we may speak of a temperature above zero as $+$, and of one below as $-$. For example, $+25°$ means $25°$ above zero, and $-10°$ means $10°$ below zero.

In navigation, north latitude is considered $+$, and south latitude $-$; west longitude is considered $+$, and east longitude $-$. For example, a place in latitude $-30°$, longitude $+95°$, would be in latitude $30°$ south of the equator, and in longitude $95°$ west of Greenwich.

Again, in financial transactions, we may consider assets as $+$, and debts or liabilities as $-$. For example, the statement that a man's property is $-\$100$, means that he owes or is in debt $100.

And in general, when we have to consider quantities the exact reverse of each other in quality or condition, we may regard quantities of either quality or condition as positive, and those of the opposite quality or condition as negative.

45. The thermometer affords an excellent illustration of the relation between positive and negative quantities.



Let OA represent the scale for temperatures above zero, and OB the scale for temperatures below zero; and let us consider the following problem:

At 7 A.M. the temperature is $-6°$; at noon it is $11°$ warmer, and at 6 P.M. it is $9°$ colder than at noon. Required the temperatures at noon and at 6 P.M.

Beginning at the scale-mark -6, and counting

11 degree-spaces upwards, we reach the scale-mark $+5$; and counting from the latter 9 degree-spaces downwards, we reach the scale-mark -4. Hence, the temperature at noon is $+5°$, and at 6 P.M. $-4°$.

EXERCISES.

46. **1.** At 7 A.M. the temperature is $-8°$; at noon it is 7° warmer, and at 6 P.M. it is 3° colder than at noon. Required the temperatures at noon and at 6 P.M.

2. A certain city was founded in the year 151 B.C., and was destroyed 203 years later. In what year was it destroyed?

3. At 7 A.M. the temperature is $+4°$; at noon it is 10° colder, and at 6 P.M. it is 6° warmer than at noon. Required the temperatures at noon and at 6 P.M.

4. What is the difference in latitude between two places whose latitudes are $+56°$ and $-31°$?

5. A man has bills receivable to the amount of \$2000, and bills payable to the amount of \$3000. How much is he worth?

6. At 7 A.M. the temperature is $-3°$, and at noon it is $+11°$. How many degrees warmer is it at noon than at 7 A.M.?

7. What is the difference in longitude between two places whose longitudes are $+25°$ and $-90°$?

8. The temperature at 6 A.M. is $-7°$, and during the morning it grows warmer at the rate of 3° an hour. Required the temperatures at 8 A.M., at 9 A.M., and at noon.

47. The *absolute value* of a quantity is the number represented by the quantity, taken independently of the sign affecting it.

Thus, the absolute value of -5 is 5.

II. ADDITION.

48. Addition, in Algebra, is the process of collecting two or more quantities into one equivalent expression, called the *sum.*

Thus, the sum of a and b is $a+b$ (Art. 8).

49. If either quantity is negative, or a polynomial, it should be enclosed in a parenthesis (Art. 20); thus,

The sum of a and $-b$ is indicated by $a+(-b)$.

The sum of $a-b$ and $c-d$ is indicated by

$$(a-b)+(c-d).$$

50. Required the sum of a and $-b$.

Using the interpretation of negative quantities as explained in Art. 44, if a man incurs a debt of \$100, we may regard the transaction either as adding $-\$100$ to his property, or as subtracting \$100 from it That is,

Adding a negative quantity is equivalent to subtracting a positive quantity of the same absolute value (Art. 47).

Thus, the sum of a and $-b$ is obtained by subtracting b from a; or,

$$a+(-b)=a-b.$$

51. It follows from Arts. 48 and 50 that the addition of monomials is effected by *uniting the quantities with their respective signs.*

Thus, the sum of a, $-b$, c, $-d$, and $-e$, is

$$a-b+c-d-e.$$

It is immaterial in what order the terms are united, provided each has its proper sign. Thus, the above result may also be expressed

$$c+a-e-d-b,$$
$$-d-b+c-e+a, \text{ etc.}$$

ADDITION OF SIMILAR TERMS.

52. **1.** Required the sum of $5a$ and $3a$.

$5a$ signifies a taken 5 times (Art. 12), and $3a$ signifies a taken 3 times. We have, therefore, a taken in all 8 times, or $8a$. That is,

$$5a+3a=8a.$$

2. Required the sum of $-5a$ and $-3a$.

$-5a$ signifies a taken 5 times *subtractively* (Art. 25), and $-3a$ signifies a taken 3 times subtractively. We have, therefore, a taken in all 8 times subtractively, or $-8a$. That is,

$$-5a-3a=-8a.$$

Therefore,

To add two similar (Art. 31) *terms of like sign, add the coefficients, affix to the result the common symbols, and prefix the common sign.*

53. **1.** Required the sum of $8a$ and $-5a$.

Since $8a$ is the sum of $3a$ and $5a$ (Art. 52, 1), the sum of $8a$ and $-5a$ is equal to the sum of $3a$, $5a$, and $-5a$, which is

$$3a+5a-5a. \qquad \text{(Art. 51.)}$$

But, by Art. 26, $5a$ and $-5a$ cancel each other, leaving the result $3a$.

Hence, $$8a+(-5a)=3a.$$

2. Required the sum of $-8a$ and $5a$.

Since $-8a$ is the sum of $-3a$ and $-5a$ (Art. 52, 2), the sum of $-8a$ and $5a$ is equal to the sum of $-3a$, $-5a$, and $5a$, which is

$$-3a-5a+5a, \text{ or } -3a.$$

Hence, $$(-8a)+5a=-3a.$$

Therefore,

To add two similar terms of unlike sign, subtract the less coefficient from the greater, affix to the result the common symbols, and prefix the sign of the greater coefficient.

Note. A clear understanding of the nature of the processes in Arts. 52 and 53 may be obtained by comparing them with the following, the negative quantities being interpreted as explained in Art. 44.

1. If a man owes \$5, and incurs a debt of \$3, he will be in debt to the amount of \$8. That is, the sum of $-\$5$ and $-\$3$ is $-\$8$.

2. If a man's assets amount to \$8, and his liabilities to \$5, he is worth \$3. That is, the sum of \$8 and $-\$5$ is \$3.

3. If a man's liabilities amount to \$8, and his assets to \$5, he is in debt to the amount of \$3. That is, the sum of $-\$8$ and \$5 is $-\$3$.

EXAMPLES.

54. Add the following:

1. 11 and -5.

2. -13 and 3.

3. 12 and -1.

4. -4 and -7.

5. $-2a$ and $7a$.

6. b and $-3b$.

7. $-11m$ and $-8m$.

8. bc and $16bc$.

9. $-2ax$ and $7ax$.

10. $-3a^2b^2$ and $-a^2b^2$.

11. $12mn^2$ and $-19mn^2$.

12. $-13abc$ and $22abc$.

13. Required the sum of $2a$, $-a$, $3a$, $-12a$, and $6a$.

Since the order of the terms is immaterial (Art. 51), we may add the positive terms first, and then the negative, and finally combine these results by the rule of Art. 53.

The sum of $2a$, $3a$, and $6a$ is $11a$.
The sum of $-a$ and $-12a$ is $-13a$.

Hence, the required sum is $11a - 13a$, or $-2a$. *Ans.*

Add the following:

14. $7a$, $-a$, and $-3a$.

15. $-6m$, m, $-11m$, and $5m$.

16. $13ab$, $-7ab$, $-8ab$, and $-6ab$.

17. $7n^2$, $-n^2$, $-3n^2$, $11n^2$, and $-10n^2$.

18. $13ax^3$, $-ax^3$, $-20ax^3$, $6ax^3$, and $-5ax^3$.

If the terms are not all similar, we may combine the similar terms, and unite the others with their respective signs.

19. Required the sum of $12a$, $-5x$, $-3y$, $-5a$, $8x$, and $-3x$.

The sum of $12a$ and $-5a$ is $7a$.
The sum of $-5x$, $8x$, and $-3x$ is 0.

Hence, the required sum is $7a-3y$. *Ans.*

Add the following:

20. $5ax$, $-11b$, $-ax$, and $6b$.

21. $2a$, $5b$, $-3c$, $-8b$, and $9c$.

22. $5m$, $-2n^2$, n, $-2m$, $-b^2$, and $3n^2$.

23. $3x$, $-y$, $-x$, 6, $-8y$, $-2x$, $4y$, and -5.

ADDITION OF POLYNOMIALS.

55. A polynomial may be regarded as the sum of its monomial terms (Art. 51). Thus, $2a-3b+4c$ is the sum of the terms $2a$, $-3b$, and $4c$.

Hence, the addition of two or more polynomials is effected by *uniting their terms with their respective signs.*

Thus, the sum of $a-b$ and $c-d$ is $a-b+c-d$.

56. Required the sum of $6a-7x$, $3x-2a+3y$, and $2x-a-mn$.

It is convenient in practice to set the expressions down one underneath the other, similar terms being in the same vertical column. Thus,

$$\begin{array}{rrrr} 6a & -\,7x & & \\ -\,2a & +\,3x & +\,3y & \\ -\;\;a & +\,2x & & -\,mn \\ \hline 3a & -\,2x & +\,3y & -\,mn, \end{array} \quad \textit{Ans.}$$

From the above principles we derive the following rule:

To add two or more expressions, set them down one underneath the other, similar terms being in the same vertical column. Find the sum of the terms in each column, and unite the results with their respective signs.

EXAMPLES.

57. Add the following:

1.

$$\begin{array}{r} 2a - 7x \\ -a + 4x \\ a + x \\ \hline \end{array}$$

2.

$$\begin{array}{r} -3ab + 2cd \\ -7ab + 8cd \\ 4ab - 6cd \\ \hline \end{array}$$

3.

$$\begin{array}{r} -11a - 5mp^2 \\ 8a + 11mp^2 \\ -9a - 7mp^2 \\ \hline \end{array}$$

4. $2a - 3b + 5c$ and $b - 5c + 2d$.

5. $9mn^2 + x^2y$, $-mn^2 + 3x^2y$, and $-6mn^2 - 7x^2y$.

6. $a^2 - 2ab + b^2$, $a^2 + 2ab + b^2$, and $2a^2 - 2b^2$.

7. $3a^2 + 2ab + 4b^2$, $5a^2 - 8ab + b^2$, and $-6a^2 + 5ab - 5b^2$.

8. $6x^3 - 7x - 4$, $x^2 - x - 2$, and $8x - 9x^2 - x^3$.

9. $4mn + 3ab - 4c$, $3x - 4ab + 2mn$, and $3m^2 - 4x$.

10. $3x - 2y - z$, $6y - 5x - 7z$, $8z - y - x$, and $4x - 9y$.

11. $6x - 3y + 7m$, $2n - x + y$, $2y - 4x - 5m - 9n$, and $m - 2x$.

12. $2x^3 - 5x^2 - x + 7$, $3x^2 - 2 - 6x^3 + 8x$, $x + 3x^3 - 4$, and $1 + 2x^2 - 5x$.

13. $2a - 3b + 4d$, $2b - 3d + 4c$, $2d - 3c + 4a + 4b$, and $2c - 3a$.

14. $2a^3 - a^2b - 2b^3$, $8a^3 - 8ab^2 - 3b^3$, $3a^2b - ab^2 + b^3$, and $6ab^2 - 2a^2b - 5a^3$.

15. $4x^3 - 10a^3 - 5ax^2 + 6a^2x$, $6a^3 + 3x^3 + 4ax^2 + 2a^2x$, $-17x^3 + 19ax^2 - 15a^2x$, and $6x^3 + 7a^2x + 5a^3 - 18ax^2$.

III. SUBTRACTION.

58. **Subtraction,** in Algebra, is the process of taking one quantity from another.

The *Subtrahend* is the quantity to be subtracted.

The *Minuend* is the quantity from which it is to be subtracted.

The *Remainder* is the result of the operation.

59. It is evident from the above that the minuend is equal to the sum of the subtrahend and the remainder.

60. Let it be required to subtract $-b$ from a.

Using the interpretation of negative quantities as explained in Art. 44, if a man cancels a debt of \$100, we may regard the transaction either as subtracting $-$\$100 from his property, or as adding \$100 to it. That is,

Subtracting a negative quantity is equivalent to adding a positive quantity of the same absolute value.

Thus, to subtract $-b$ from a, we add b to a; or

$$a-(-b)=a+b.$$

Hence, *to subtract one quantity from another, change the sign of the subtrahend, and add the result to the minuend.*

61. **1.** Subtract $5a$ from $2a$.

By Art. 60, the result is equal to the sum of $-5a$ and $2a$, which is $-3a$.

2. From $-2a$ subtract $5a$.

The result is equal to the sum of $-2a$ and $-5a$, or $-7a$.

3. From $5a$ take $-2a$.

Result, $5a+2a$, or $7a$.

4. From $-2a$ take $-5a$.

Result, $-2a+5a$, or $3a$.

EXAMPLES.

62. Subtract the following:

1. -3 from 11. **3.** -8 from -3. **5.** 23 from 10.

2. 16 from -5. **4.** -11 from -17. **6.** -13 from 11.

7.	**8.**	**9.**	**10.**	**11.**
$27a$	$17x$	$-13y$	$-10mn$	$5a^2b$
$\underline{13a}$	$\underline{-11x}$	$\underline{4y}$	$\underline{-18mn}$	$\underline{14a^2b}$

12. From $9ab$ take $-2ab$.

13. From xy take $-cd$.

14. From $17m^3$ take $41m^3$.

15. From $-5x$ take 3.

16. From $-x^2y^2$ take $5x^2y^2$.

17. From $-70abc$ take $-52abc$.

18. From $-7m^2$ take $-8n^2$.

19. From $-33x^3y^2$ take $19x^3y^2$.

20. From $5ab$ take the sum of $9ab$ and $-2ab$.

21. From the sum of $-11x^3$ and $8x^3$ take the sum of $-10x^3$ and $4x^3$.

SUBTRACTION OF POLYNOMIALS.

63. When the subtrahend is a polynomial, each of its terms is to be subtracted from the minuend. Hence,

To subtract one polynomial from another, change the sign of each term of the subtrahend, and add the result to the minuend.

It will be found convenient to place the subtrahend under the minuend, similar terms being in the same vertical column.

64. **1.** Subtract $5x^2y - 3ab + m^2$ from $3x^2y - 2ab + 4n$.

Changing the sign of each term of the subtrahend, and adding the result to the minuend, we have

$$\begin{array}{rrrr} 3x^2y & -2ab & +4n & \\ -5x^2y & +3ab & & -m^2 \\ \hline -2x^2y & +ab & +4n & -m^2, \end{array} \quad \textit{Ans.}$$

Note. The student should endeavor to perform *mentally* the operation of changing the sign of each term in the subtrahend, as shown in the following example:

2. From $5a^3 - 7b^3 - 2a^2b$ subtract $3a^2b - 4ab^2 - 2b^3 + a^3$.

$$\begin{array}{r} 5a^3 - 2a^2b \qquad\qquad - 7b^3 \\ a^3 + 3a^2b - 4ab^2 - 2b^3 \\ \hline 4a^3 - 5a^2b + 4ab^2 - 5b^3, \end{array}\ \textit{Ans.}$$

EXAMPLES.

Subtract the following:

3.

$$\begin{array}{r} ab + cd - ax \\ 4ab - 3cd - 4ax \\ \hline \end{array}$$

4.

$$\begin{array}{r} 7x + 5y - 3a \\ x - 7y + 5a - 4 \\ \hline \end{array}$$

5. From $a - b + c$ take $a + b - c$.

6. From $a^2 + 2ab + b^2$ take $a^2 - 2ab + b^2$.

7. From $7abc - 11x + 5y - 48$ take $11abc + 3x + 7y + 100$.

8. Subtract $3m + y^2 - 5a - 7$ from $5m - 3y^2 + 7a - 6$.

9. Subtract $17x^2 + 5y^2 - 4ab + 7$ from $31x^2 - 3y^2 + ab$.

10. From $6a + 3b - 5c + 1$ subtract $6a - 3b - 5c$.

11. From $3m - 5n + r - 2s$ take $2r + 3n - m - 5s$.

12. Take $4a - b + 2c - 5d$ from $d - 3b + a - c$.

13. From $m^2 + 3n^3$ subtract $-4m^2 - 6n^3 + 71x$.

14. From $4c - 3b - 5d + 2x$ take $3a + 8d - b - 6c$.

15. From $a - b - c$ take the sum of $-2a + b + c$ and $a - b + c$.

16. From $x^4 + 2x^3 - 3x + 4$ take $3x^3 + 3x^2 + 5x - 7$.

17. From $4a^3 - 3ab^2 - 5b^3$ subtract $6a^2b - ab^2 + 4b^3$.

18. From $a^2 - 8 + 2a^4 - 3a^3$ take $6a - 11 - 5a^2 - 2a^4$.

19. Take $2x^2 - y^2$ from the sum of $x^2 - 2xy + 3y^2$ and $xy - 4y^2$.

20. From the sum of $x + 2y - 3z$ and $3y - 4x + z$ take $z - 5x + 5y$.

21. From $7a^3 + 3 - 5a^4 + a - 5a^2$ subtract $2a - 6a^2 - 2a^3 + 9 - 11a^4$.

22. From $-7y^3 + 3x^2y - 2x^3 + 6xy^2$ subtract $8x^2y - 2xy^2 + x^3 - 9y^3$.

23. From the sum of $2x^3 - x + 5$ and $x^2 + 8x - 11$ take the sum of $x^3 - 9x^2 - 11x$ and $-4x^3 + 3x^2 - 6$.

24. From the sum of $a^2 + ab + b^2$ and $a^2 - 4ab + 5b^2$ take the sum of $4a^2 + 7b^2 - 2ab$ and $3ab - a^2 - 2b^2$.

25. From $3x^2 - 7y - 2 + xy - 5y^2$ subtract $-5xy + 6x - 2x^2 - 8 + 2y^2$.

26. From $3x^5 - 8x^4 + 3x^3 - 5x^2 - 2x$ subtract $-3x^4 + 4x^3 + 6x^2 - 6x + 2$.

27. From the sum of $2x^3 - x^2y - 5xy^2$ and $3x^2y - 5xy^2 - 4y^3$ take the sum of $-2x^3 - 7x^2y - 6y^3$ and $-6xy^2 + 5y^3$.

28. From the sum of $a^4 - 1$ and $2a^3 - 10a^2 - 7a$ subtract the sum of $-3a^4 + 2a^2 - 5a$ and $-5a^3 - 12a^2 + 3$.

Note. In Arithmetic, addition always implies *augmentation*, and subtraction *diminution*. In Algebra this is not always the case; for example, in adding -2 to 5, the sum is 3, which is less than 5. Again, in subtracting -2 from 5, the remainder is 7, which is greater than 5.

Thus the terms *Addition*, *Subtraction*, *Sum*, and *Remainder* have a much more general signification in Algebra than in Arithmetic.

IV. USE OF PARENTHESES.

65. The use of parentheses (Art. 20) is very frequent in Algebra, and it is necessary to have rules for their removal or introduction.

66. The expression

$$2a-3b+(5b-c+2d)$$

indicates that the quantity $5b-c+2d$ is to be added to $2a-3b$. If the addition be performed, we obtain (Art. 55)

$$2a-3b+5b-c+2d.$$

Again, the expression

$$2a-3b-(5b-c+2d)$$

indicates that the quantity $5b-c+2d$ is to be subtracted from $2a-3b$. If the subtraction be performed, we obtain (Art. 63)

$$2a-3b-5b+c-2d.$$

67. It will be observed that in the first case the signs of the terms within the parenthesis are *unchanged* when the parenthesis is removed; while in the second case the sign of each term within is *changed*, from $+$ to $-$, or from $-$ to $+$.

We have then the following rule for removing a parenthesis:

A parenthesis preceded by a $+$ sign may be removed without altering the signs of the enclosed terms.

A parenthesis preceded by a $-$ sign may be removed, if the sign of each enclosed term be changed, from $+$ to $-$, or from $-$ to $+$.

68. Since the brackets, the braces, and the vinculum (Art. 20) have the same signification as the parenthesis, the rule for their removal is the same.

It should be observed in the case of the vinculum that the sign apparently prefixed to the first term underneath, is in reality the sign of the vinculum. Thus, $+\overline{a-b}$ and $-\overline{a-b}$ are equivalent to $+(a-b)$ and $-(a-b)$, respectively.

EXAMPLES.

69. **1.** Remove the parentheses from

$$2a-3b-(5a-4b)+(4a-b).$$

By the rule of Art. 67, the expression becomes

$$2a-3b-5a+4b+4a-b=a, \textit{ Ans.}$$

Parentheses are often found enclosing others. In this case they may be removed in succession by the rule of Art. 67, and it is better to remove first the *innermost* pair.

2. Simplify the expression

$$4x-\{3x+(-2x-\overline{x-a})\}.$$

We remove the vinculum first, and the others in succession. Thus,

$$\begin{aligned}&4x-\{3x+(-2x-\overline{x-a})\}\\ =&4x-\{3x+(-2x-x+a)\}\\ =&4x-\{3x-2x-x+a\}\\ =&4x-3x+2x+x-a=4x-a, \textit{ Ans.}\end{aligned}$$

Reduce the following expressions to their simplest forms by removing the parentheses, etc., and uniting similar terms:

3. $a-(b-c)+(-d+e)$.

4. $5x-\{2x-3y\}-[-2x+4y]$.

5. $a-b+c-\overline{a+b-c}-\overline{c-b-a}$.

6. $m^2-2n+\{a-n+3m^2\}-\overline{5a+3n-m^2}$.

7. $a^2-b^2-(a^2-2ab+b^2)-[a^2+2ab+b^2]$.

8. $3a-(2a-\{a+2\})$.

9. $a-(b+\{-c+d\}-e)$.

10. $a-[(-b+c)-(d-e)]$.

11. $3x-[2y+\overline{x-y}]+[3y-\overline{2x+y}]$.

12. $14x-(5x-9)-\{4-3x-(2x-3)\}$.

13. $2m-[n-\{3m-(2n-m)\}]$.

14. $3x-(5x+[-4x-\overline{y-x}])-(-x-3y)$.

15. $3c+(2a-[5c-\{3a+\overline{c-4a}\}])$.

16. $5a-(4a-\{-3a-[2a-\overline{a-1}]\})$.

17. $8x-[5x-(3x-4)-\{7x+(-9x+2)\}]$.

18. $2m-[3m-\{m-(2m-\overline{3m+4})\}-(5m-2)]$.

19. $c-[2c-(6a-b)-\{c-(5a+2b)-(a-3b)\}]$.

20. $3a-\{b-[b-(a+b)-\{-b-(b-\overline{a-b})\}]\}$.

70. To enclose any number of terms in a parenthesis, we take the converse of the rule of Art. 67:

Any number of terms may be enclosed in a parenthesis preceded by a $+$ sign, without altering their signs.

Any number of terms may be enclosed in a parenthesis preceded by a $-$ sign, if the sign of each term be changed, from $+$ to $-$, or from $-$ to $+$.

71. 1. Enclose the last three terms of $a-b+c-d+e$ in a parenthesis preceded by a $-$ sign.

Result, $a-b-(-c+d-e)$.

In each of the following expressions, enclose the last three terms in a parenthesis preceded by a $-$ sign:

2. $a+b+c+d$.

3. $3a-2b+5c-4d$.

4. m^3+5m^2-6m+3.

5. $x^3y-x^2y^2-xy^3+y^4$.

6. $x^4-3x^3+2x^2-5x-8$.

7. $a^2-b^2-c^2+2ab+2ac$.

8. In each of the above results, enclose the last two terms in an *inner* parenthesis preceded by a $-$ sign.

V. MULTIPLICATION.

72. Multiplication, in Algebra, is the process of taking one quantity as many times as there are units in another.

Thus, the multiplication of a by b, which is expressed ab (Art. 10), signifies that the quantity a is to be taken b times.

73. The *Multiplicand* is the quantity to be multiplied or taken.

The *Multiplier* is the quantity which shows how many times it is to be taken.

The *Product* is the result of the operation.

The multiplicand and multiplier are called *factors.*

74. In Arithmetic, the product of two numbers is the same in whatever order they are taken; thus, we have 3×4 or 4×3, each equal to 12.

Similarly, in Algebra, we have $a \times b$ or $b \times a$, each equal to ab.

That is, *the product of the factors is the same in whatever order they are taken.*

75. Required the product of c and $a - b$.

In Arithmetic, if we wish to multiply 87 by 98, we may express the multiplier in the form $100 - 2$; we should then multiply 87 by 100, and afterwards by 2, and subtract the second result from the first.

Similarly, in Algebra, to multiply c by $a - b$, we should multiply c by a, and afterwards by b, and subtract the second result from the first. Thus, the required product is

$$ac - bc.$$

76. Required the product of $a - b$ and $c - d$.

As in Art. 75, we should first multiply $a - b$ by c, and

afterwards by d, and subtract the second result from the first.

The product of $a-b$ and c is $ac-bc$ (Art. 75).
The product of $a-b$ and d is $ad-bd$.

Subtracting the second result from the first, the required product is

$$ac-bc-ad+bd.$$

77. We observe, in the preceding article, that the product is formed by multiplying each term of the multiplicand by each term of the multiplier, with the following results in regard to signs:

The product of the terms $+a$ and $+c$ gives the term $+ac$.
The product of the terms $-b$ and $+c$ gives the term $-bc$.
The product of the terms $+a$ and $-d$ gives the term $-ad$.
The product of the terms $-b$ and $-d$ gives the term $+bd$.

From these considerations we may state what is called the **Rule of Signs** in Multiplication, as follows:

$+$ *multiplied by* $+$, *and* $-$ *multiplied by* $-$, *produce* $+$;
$+$ *multiplied by* $-$, *and* $-$ *multiplied by* $+$, *produce* $-$.

Or, as it is usually expressed with regard to the product of two terms,

Like signs produce $+$, *and unlike signs produce* $-$.

78. Required the product of $7a$ and $2b$.

Since the factors may be written in any order (Art. 74), we have

$$7a\times 2b=7\times 2\times a\times b=14ab.$$

That is, *the coefficient of the product is the product of the coefficients of the factors.*

79. Required the product of a^3 and a^2.

By Art. 13, $a^3=a\times a\times a$, and $a^2=a\times a$. Hence,

$$a^3\times a^2=a\times a\times a\times a\times a=a^5.$$

That is, *the exponent of a letter in the product is the sum of its exponents in the factors.*

Thus, $$a^5 \times a^3 \times a = a^{5+3+1} = a^9.$$

MULTIPLICATION OF MONOMIALS.

80. We derive from Arts. 77, 78, and 79 the following rule for the product of two monomials:

To the product of the coefficients annex the literal quantities, giving to each letter an exponent equal to the sum of its exponents in the factors. Make the product + when the factors have the same sign, and − when they have different signs.

EXAMPLES.

1. Multiply $2a^5$ by $7a^4$.

By the rule, $2a^5 \times 7a^4 = 14a^{5+4} = 14a^9$, *Ans.*

2. Multiply a^3b^2c by $-5a^2bd$.

$$a^3b^2c \times (-5a^2bd) = -5a^5b^3cd, \text{ Ans.}$$

3. Multiply $-7x^m$ by $5x^3$.

$$-7x^m \times 5x^3 = -35x^{m+3}, \text{ Ans.}$$

4. Multiply $-11x^m$ by $-8x^m$.

$$-11x^m \times (-8x^m) = 88x^{2m}, \text{ Ans.}$$

Multiply the following:

5. 13 by -19.

6. -18 by 12.

7. -22 by -51.

8. $15m^6n^6$ by $3mn$.

9. $17abc$ by $-8abc$.

10. $-17a^4c^2$ by $3a^2c^2$.

11. $-11n^2y$ by $-5n^6z$.

12. $-6a^2bc$ by a^3bm.

13. $-12a^2x$ by $-2x^2y$.

14. $-2a^mb^n$ by $5a^3b^n$.

15. $3a^3x^5y^2$ by $11ax^4y^5$.

16. $3a^mb^n$ by $-5a^nb^r$.

It is evident from the Rule of Signs (Art. 77) that the product of three negative terms is negative; of four negative terms, positive; and so on.

Hence the product of three or more monomials will be positive or negative, according as the number of *negative* factors is odd or even.

17. Required the product of $-2a^2b^3$, $6bc^5$, and $-7c^2d$.

$$-2a^2b^3 \times 6bc^5 \times -7c^2d = 84a^2b^4c^7d, \text{ Ans.}$$

In this case the product is positive, as there are two negative factors.

Multiply the following:

18. $5a$, $-6b$, and $7c$.

19. $-2a^2$, $-11a^3$, and $-9a$.

20. $-3ab^2$, $-2bc^2$, and $7cd^2$.

21. $4x^my^n$, $-x^ny^nz^5$, and $15y^2z^r$.

22. $-2a$, $-3a^2$, $-4a^3$, and $-5a^4$.

23. $-a^2bc$, $2b^2cd$, $-5a^3cd$, and $-3ab^5d^4$.

24. $-7m^nx^2$, $m^nx^ry^2$, $2x^3$, and $-8my^{2r}$.

25. $6xy^2$, $-x^3z$, $3y^4z^2$, $-2xz^5$, and $-4yz$.

MULTIPLICATION OF POLYNOMIALS BY MONOMIALS.

81. In Art. 75 we showed that the product of $a-b$ and c was $ac-bc$. We have then the following rule for the product of a polynomial by a monomial:

Multiply each term of the multiplicand by the multiplier, and connect the results with their proper signs.

EXAMPLES.

1. Multiply $2x^2-5x-7$ by $8x^3$.

By the rule, the product is $16x^5-40x^4-56x^3$, *Ans.*

2. Multiply $-5ab^4$ by $3a^2b-4ab^3$.

$$\begin{array}{r} 3a^2b-4ab^3 \\ -\ \ 5ab^4 \\ \hline -15a^3b^5+20a^2b^7, \end{array}\ \textit{Ans.}$$

Multiply the following:

3. $3x-5$ by $4x$.

4. a^2b+ab^2 by $-ab$.

5. $8a^2bc-d$ by $5ad^2$.

6. x^2-2x-3 by $-4x$.

7. $-2x^3$ by $3x^2+6x-7$.

8. m^2+mn+n^2 by m^2n^2.

9. $-2m$ by $3m^2-5mn-n^2$.

10. $-x^4-10x^3+5$ by $-2x^3$.

11. $a^2+13ab-6b^2$ by $4ab^2$.

12. $-6a^2c$ by $5-6ac-8a^3$.

13. $5x^3-4x^2-3x+2$ by $-6x^5$.

14. a^2b^2 by $a^3-3a^2b+3ab^2-b^3$.

MULTIPLICATION OF POLYNOMIALS BY POLYNOMIALS.

82. In Art. 76 it was shown that the product of $a-b$ and $c-d$ was $ac-bc-ad+bd$. We have then the following rule for the product of two polynomials:

Multiply each term of the multiplicand by each term of the multiplier, and add the partial products.

EXAMPLES.

1. Multiply $3a-2b$ by $2a-5b$.

In accordance with the rule, we multiply $3a-2b$ by $2a$, and then by $-5b$, and add the partial products. A convenient arrangement of the work is shown below, similar terms being in the same vertical column.

$$\begin{array}{lll} 3a & -\ \ 2b & \\ 2a & -\ \ 5b & \\ \hline 6a^2 & -\ \ 4ab & \\ & -15ab & +10b^2 \\ \hline 6a^2 & -19ab & +10b^2, \end{array}\ \textit{Ans.}$$

2. Multiply x^2+1-x^3-x by $x+1$.

It is convenient to have both multiplicand and multiplier arranged in the same order of powers (Art. 36), and to write the partial products in the same order.

Arranging the expressions according to the ascending powers of x, we have

$$\begin{array}{llll}
1-x+x^2-x^3 \\
1+x \\
\hline
1-x+x^2-x^3 \\
\quad\; x-x^2+x^3-x^4 \\
\hline
1 \qquad\qquad\qquad -x^4, \textit{ Ans.}
\end{array}$$

3. Multiply $6ab-8b^2+4a^2$ by $-4b^2+2a^2-3ab$.

Arranging according to the descending powers of a, we have

$$\begin{array}{lllll}
4a^2+6ab-8b^2 & & & \\
2a^2-3ab-4b^2 & & & \\
\hline
8a^4+12a^3b & -16a^2b^2 & & \\
\quad\;\; -12a^3b & -18a^2b^2 & +24ab^3 & \\
 & -16a^2b^2 & -24ab^3 & +32b^4 \\
\hline
8a^4 & -50a^2b^2 & & +32b^4, \textit{ Ans.}
\end{array}$$

Note. The correctness of the answers may be tested by working the examples with the multiplicand and multiplier interchanged.

Multiply the following:

4. $3x+2$ and $5x-7$.

5. $6x-5$ and $3-2x$.

6. $3a-2b$ and $-2a+4b$.

7. $3-5xy$ and $-6-10xy$.

8. a^2+ab+b^2 and $b-a$.

9. $2a^2b-3ab^2$ and $5a^2b+6ab^2$.

10. $1+x+x^2+x^3$ and $ax-a$.

11. $3x^2-2xy-y^2$ and $2x-4y$.

12. $m^2-mn-3n^2$ and $2m^2-6mn$.

13. x^2+2x+1 and x^2-2x+3.

14. $5a^2 + 4b^2 - 3ab$ and $6a - 5b$.

15. $4x^3 + 6x - 7$ and $2x^2 - 3$.

16. $a + b - c$ and $a - b + c$.

17. $2x^2 - 3x + 5$ and $x^2 + x - 1$.

18. $3x^2 - 7x + 4$ and $2x^2 + 9x - 5$.

19. $2x^3 - 3x^2 - 5x - 1$ and $3x - 5$.

20. $6m - 2m^2 - 5 - m^3$ and $m^2 + 10 - 2m$.

21. $2x^3 + 5x^2 - 8x - 7$ and $4 - 5x - 3x^2$.

22. $a^3b - a^2b^2 - 4ab^3$ and $2a^2b - ab^2$.

23. $x^{m+2}y - 3xy^{n-1}$ and $4x^{m+5}y^2 - 4x^4y^n$.

24. $x^2 + y^2 - xy$ and $xy + y^2 + x^2$.

25. $2ab + b^2 + 4a^2$ and $4a^2 - 2ab + b^2$.

26. $6x^4 - 3x^3 - x^2 + 6x - 2$ and $2x^2 + x + 2$.

27. $m^4 - m^3n + m^2n^2 - mn^3 + n^4$ and $m^2 - 2mn - 3n^2$.

28. $27x^3 + 9x^2y + 3xy^2 + y^3$ and $9x^2 - 6xy + y^2$.

29. $a^3 - 3a^2b + 3ab^2 - b^3$ and $a^2 - 2ab + b^2$.

30. $x^2 + y^2 + z^2 - xy - yz - zx$ and $x + y + z$.

31. $2x^3 - 3x^2 + 5x - 1$ and $3x^3 - x^2 - 2x - 5$.

32. $ab + cd + ac + bd$ and $ab + cd - ac - bd$.

33. $2a^3 - 5a^2 - 6a + 4$ and $4a^3 + 10a^2 - 12a - 8$.

Find the product of the following:

34. $x - 3$, $x + 4$, and $x - 7$.

35. $a + b$, $a^2 - ab + b^2$, and $a^3 - b^3$.

36. $2m - 1$, $3m + 4$, and $6m - 5$.

37. $x + 1$, $3x - 2$, and $3x^2 - x - 2$.

38. $x^2 + x + 1$, $x^2 - x + 1$, and $x^4 - x^2 + 1$.

39. $a+b$, $a-b$, a^2+b^2, and a^4+b^4.

40. $m+1$, $m-1$, $m+2$, and $m-2$.

41. $2x-1$, $3x+2$, $4x-3$, and $5x+4$.

42. $a+b$, $a-b$, $a+2b$, and $a^3-2a^2b-ab^2+2b^3$.

83. The product of two or more polynomials may be *indicated* by enclosing each of them in a parenthesis, and writing them one after the other.

Thus, the product of $x+2$, $x-3$, and $2x-7$ is indicated by

$$(x+2)(x-3)(2x-7).$$

Similarly, the expression $(a+b+c)^2$ indicates that $a+b+c$ is to be multiplied by itself (Art. 13).

When the operations indicated are performed, the expression is said to be *expanded* or *simplified*.

EXAMPLES.

1. Simplify the expression $(a-2x)^2-2(x+3a)(a-x)$.

To simplify the expression, we should expand $(a-2x)^2$ and $2(x+3a)(a-x)$, and subtract the second result from the first.

$$(a-2x)^2=a^2-4ax+4x^2$$
$$2(x+3a)(a-x)=6a^2-4ax-2x^2$$

Subtracting the second result from the first, we have

$$a^2-4ax+4x^2-6a^2+4ax+2x^2=6x^2-5a^2, \textit{ Ans.}$$

Simplify the following:

2. $(a+b+c+d)^2$.

3. $(a-b)(c-d)+(a-c)(b-d)$.

4. $(2x-3)^2+(1-x)(3x-9)$.

5. $(a+b+c)^2-(a-b+c)^2$.

6. $(2a-5b)^2-4(a-2b)(a-3b)$.

7. $(a-b)^2(a+b)^2$.

8. $(1+x)(1+x^4)(1-x+x^2-x^3)$.

9. $(1+a)^3-(1-a)(1+a^2)$.

10. $[x-(2y+3z)][x-(2y-3z)]$.

11. $(x+y)(x^3-y^3)[x^2-y(x-y)]$.

12. $(a+b)(b+c)-(c+d)(d+a)-(a+c)(b-d)$.

13. $(a+b+c)^2+(a-b-c)^2+(b-c-a)^2+(c-a-b)^2$.

14. $(a-b)(b-c)+(b-c)(c-a)+(c-a)(a-b)$.

15. $x(x-2y)+y(y-2z)+z(z-2x)-(x-y-z)^2$.

16. $x(x+1)(x+2)(x+3)+1-(x^2+3x+1)^2$.

17. $(a+b+c)^2-(a-b-c)^2+(b-c-a)^2-(c-a-b)^2$.

18. $[(m+2n)^2-(2m-n)^2][(2m+n)^2-(m-2n)^2]$.

19. $(x+y+z)^3-(x^3+y^3+z^3)-3(y+z)(z+x)(x+y)$.

84. Since $(+a)(+b)=ab$, and $(-a)(-b)=ab$, it follows that in the indicated product of two factors *all the signs of both factors may be changed without altering the value of the expression.* Thus,

$$(a-b)(c-d) \text{ is equal to } (b-a)(d-c).$$

Similarly, we may show that in the indicated product of any number of factors, *the signs of any even number of factors may be changed without altering the value of the expression.*

Thus, $(a-b)(c-d)(e-f)$, by changing the signs of the second and third factors, may be written in the equivalent form $(a-b)(d-c)(f-e)$.

VI. DIVISION.

85. Division, in Algebra, is the process of finding one of two factors, when their product and the other factor are given.

Hence, Division is the converse of Multiplication.

Thus, the division of $14ab$ by $7a$, which is expressed $\frac{14ab}{7a}$ (Art. 15), signifies that we are to find a quantity which, when multiplied by $7a$, will produce $14ab$.

86. The *Dividend* is the product of the factors.
The *Divisor* is the given factor.
The *Quotient* is the required factor.

87. Since the dividend is the product of the divisor and quotient, it follows, from Art. 77, that:

If the divisor is $+$, and the quotient is $+$, the dividend is $+$.
If the divisor is $-$, and the quotient is $+$, the dividend is $-$.
If the divisor is $+$, and the quotient is $-$, the dividend is $-$.
If the divisor is $-$, and the quotient is $-$, the dividend is $+$.

In other words, if the dividend and divisor are both $+$, or both $-$, the quotient is $+$; and if the dividend and divisor are one $+$, and the other $-$, the quotient is $-$. Hence, in Division as in Multiplication,

Like signs produce $+$, *and unlike signs produce* $-$.

88. Required the quotient of $14ab$ divided by $7a$.

By Art 85, we are to find a quantity which, when multiplied by $7a$, will produce $14ab$. That quantity is evidently $2b$; hence

$$\frac{14ab}{7a} = 2b.$$

That is, *the coefficient of the quotient is the coefficient of the dividend, divided by the coefficient of the divisor.*

89. Required the quotient of a^5 divided by a^3.

We are to find a quantity which, when multiplied by a^3, will produce a^5. That quantity is evidently a^2; hence

$$\frac{a^5}{a^3} = a^2.$$

That is, *the exponent of a letter in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.*

For example, $$\frac{a^m}{a^n} = a^{m-n}.$$

DIVISION OF MONOMIALS.

90. We derive from Arts. 87, 88, and 89 the following rule for the division of monomials:

To the quotient of the coefficients annex the literal quantities, giving to each letter an exponent equal to its exponent in the dividend minus its exponent in the divisor. Make the quotient + when the dividend and divisor have like signs, and − when they have unlike signs.

EXAMPLES.

1. Divide $54a^7$ by $-9a^4$.

By the rule, $$\frac{54a^7}{-9a^4} = -6a^{7-4} = -6a^3, \text{ Ans.}$$

2. Divide $-2a^3b^2cd^4$ by abd^4.

$$\frac{-2a^3b^2cd^4}{abd^4} = -2a^2bc, \text{ Ans.}$$

Note. A literal quantity having the same exponent in the dividend and divisor, as d^4 in Ex. 2, is *canceled* by the operation of division, and does not appear in the quotient.

3. Divide $-91x^my^nz^r$ by $-13x^ny^nz^3$.

$$\frac{-91x^my^nz^r}{-13x^ny^nz^3} = 7x^{m-n}z^{r-3}, \text{ Ans.}$$

Divide the following:

4. 84 by -12.
5. -343 by 7.
6. -324 by -18.
7. 444 by -37.
8. $12a^5$ by $4a$.
9. $-a^2c$ by ac.
10. $2m^3n^4$ by $-mn^3$.
11. $-8x^2y^2$ by $-4x^2$.
12. $30a^5b^3$ by $5a^2b$.
13. $14m^3n^4$ by $-7mn^3$.
14. $-18x^2y^5z$ by $9x^2z$.
15. $-65a^3b^3c^3$ by $-5ab^2c^3$.
16. $72m^5n$ by $-12m^2$.
17. $12x^ay^b$ by $3x^cy^d$.
18. $-18a^mb$ by $6ab$.
19. $-144c^5d^7e^6$ by $-36c^2d^3e$.
20. $-3a^{m+2}$ by a^{m+1}.
21. $a^{m+n}b^{m+n}$ by a^mb^n.
22. $-91x^4y^3z^2$ by $-13x^3y^2$.
23. $18m^3n^4p^5$ by $-2m^2np^5$.

DIVISION OF POLYNOMIALS BY MONOMIALS.

91. The operation being simply the converse of Art. 81, we have the following rule:

Divide each term of the dividend by the divisor, and connect the results with their proper signs.

EXAMPLES.

1. Divide $9a^3b - 6a^4c + 12a^2bc$ by $-3a^2$.

By the rule,

$$\frac{9a^3b - 6a^4c + 12a^2bc}{-3a^2} = -3ab + 2a^2c - 4bc, \textit{ Ans.}$$

Divide the following:

2. $8a^3bc + 16a^5bc - 4a^7c^2$ by $4a^2c$.
3. $9x^4 + 27x^3 - 21x^2$ by $-3x^2$.
4. $30a^3 - 75a^4b$ by $15a^3$.
5. $2x^3y^2z - 12xy^2z^3$ by $-2xy^2z$.

6. $5a^2bc - 5ab^2c + 5abc^2$ by $-5abc$.

7. $4x^7 - 8x^6 - 14x^5 + 2x^4 - 6x^3$ by $2x^3$.

8. $-12a^pb^q - 30a^{12}b^5 + 108a^nb^n$ by $-6a^mb^n$.

9. $20x^4 - 12x^2 - 28x$ by $4x$.

10. $-a^2b^2c - ab^2c^2 + a^2bc^2$ by $-abc$.

11. $9a^5bc - 3a^2b + 18a^3bc$ by $-3ab$.

12. $15x^my^nz^r - 35x^{m+2}y^{2n}z$ by $5x^my^nz$.

13. $20a^4bc + 15abd^3 - 10a^2b$ by $-5ab$.

DIVISION OF POLYNOMIALS BY POLYNOMIALS.

92. Required the quotient of $12 + 10x^3 - 11x - 21x^2$ divided by $2x^2 - 4 - 3x$.

Arranging both dividend and divisor according to the descending powers of x (Art. 37), we are to find a quantity which, when multiplied by the divisor, $2x^2 - 3x - 4$, will produce $10x^3 - 21x^2 - 11x + 12$.

It is evident, from Art. 82, that the term containing the highest power of x in the product, is the product of the terms containing the highest powers of x in the factors. Hence $10x^3$ is the product of $2x^2$ and the term containing the highest power of x in the quotient. Therefore the term containing the highest power of x in the quotient is $10x^3$ divided by $2x^2$, or $5x$.

Multiplying the divisor by $5x$, we have the product $10x^3 - 15x^2 - 20x$; which, when subtracted from the dividend, leaves the remainder $-6x^2 + 9x + 12$.

This remainder is the product of the divisor by the rest of the quotient; hence, to obtain the next term of the quotient, we proceed as before, regarding $-6x^2 + 9x + 12$ as a new dividend. Dividing the term containing the highest power of x, $-6x^2$, by the term containing the highest power

of x in the divisor, $2x^2$, we have -3 as the second term of the quotient.

Multiplying the divisor by -3, we have $-6x^2+9x+12$; which, when subtracted from the second dividend, leaves no remainder. Hence $5x-3$ is the required quotient.

It is customary to arrange the work as follows:

$$\begin{array}{r|ll}
10x^3-21x^2-11x+12 & 2x^2-3x-4, & \textit{Divisor.} \\
\underline{10x^3-15x^2-20x} & 5x\ -3, & \textit{Quotient.} \\
-\ 6x^2+\ 9x+12 & & \\
\underline{-\ 6x^2+\ 9x+12} & &
\end{array}$$

Note. We might have solved the example by arranging the dividend and divisor according to the *ascending* powers of x, in which case the quotient would have appeared in the form $-3+5x$.

93. From Art. 92, we derive the following rule for the division of polynomials:

Arrange both dividend and divisor in the same order of powers of some common letter.

Divide the first term of the dividend by the first term of the divisor, giving the first term of the quotient.

Multiply the whole divisor by this term, and subtract the product from the dividend, arranging the remainder in the same order of powers as the dividend and divisor.

Regard the remainder as a new dividend, and proceed as before; continuing until there is no remainder.

Note. The work may be verified by multiplying the quotient by the divisor, which should of course give the dividend.

EXAMPLES.

1. Divide $21x^2y^2-22xy-8$ by $3xy-4$.

$$\begin{array}{r|l}
21x^2y^2-22xy-8 & 3xy-4 \\
\underline{21x^2y^2-28xy} & 7xy+2, \quad \textit{Ans.} \\
6xy-8 & \\
\underline{6xy-8} &
\end{array}$$

2. Divide $8+18x^4-56x^2$ by $-6x^2+4+8x$.

Arranging according to the ascending powers of x,

$$
\begin{array}{rrrrl|l}
8 & -56x^2 & +18x^4 & & & 4+8x-6x^2 \\
\cline{6-6}
8 & +16x & -12x^2 & & & 2-4x-3x^2,\ \textit{Ans.} \\
\cline{1-3}
 & -16x & -44x^2 & +18x^4 & & \\
 & -16x & -32x^2 & +24x^3 & & \\
\cline{2-4}
 & & -12x^2 & -24x^3 & +18x^4 & \\
 & & -12x^2 & -24x^3 & +18x^4 & \\
\cline{3-5}
\end{array}
$$

3. Divide $9ab^2+a^3-9b^3-5a^2b$ by $3b^2+a^2-2ab$.

Arranging according to the descending powers of a,

$$
\begin{array}{l}
a^2-2ab+3b^2)\,a^3-5a^2b+9ab^2-9b^3\,(a-3b,\ \textit{Ans.} \\
\qquad\qquad\qquad\ \underline{a^3-2a^2b+3ab^2} \\
\qquad\qquad\qquad\qquad\ -3a^2b+6ab^2-9b^3 \\
\qquad\qquad\qquad\qquad\ \underline{-3a^2b+6ab^2-9b^3}
\end{array}
$$

Divide the following:

4. $6x^2-x-35$ by $3x+7$.

5. $2-3ax-2a^2x^2$ by $1-2ax$.

6. $a^2-4ab+4b^2$ by $a-2b$.

7. $59x-56-15x^2$ by $3x-7$.

8. $3b^3+3ab^2-4a^2b-4a^3$ by $b+a$.

9. $2a^3x-2ax^3$ by $ax-a^2$.

10. $18x^3-5x+1$ by $6x^2+2x-1$.

11. $8m^3+35-36m$ by $5+2m$.

12. $27x^3+y^3$ by $3x+y$.

13. $16m^4-1$ by $2m-1$.

14. $a^2-b^2+c^2-2ac$ by $a+b-c$.

15. $8a^3+36a^2b+54ab^2+27b^3$ by $2a+3b$.

16. $x^4+y^4+x^2y^2$ by x^2+y^2+xy.

17. $2x^4 - 19x^2 + 9$ by $2x^3 + 6x^2 - x - 3$.

18. $8m^3 + 3n^3 - 4m^2n - 6mn^2$ by $2m - n$.

19. $4x^4 - 8x^3 - 6x^2 + 24$ by $2x - 4$.

20. $23x^2 - 48 + 6x^4 - 2x - 31x^3$ by $6 + 3x^2 - 5x$.

21. $4a^5 + 27 - a^3$ by $9 - 3a^3 + 4a^2 + 2a^4 - 6a$.

22. $x^4 - 9x^2 - 6xy - y^2$ by $x^2 + 3x + y$.

23. $a^8 - 81b^4$ by $a^2 + 3b$.

24. $x^2 - y^2 + 2yz - z^2$ by $x + y - z$.

25. $3x^4 - 14x^2 + 8$ by $x - 2$.

26. $y^6 + x^5y$ by $x + y$.

27. $15m^4 + 50m^2 + 15 - 32m - 32m^3$ by $3m^2 + 5 - 4m$.

28. $1 + 4x^3 + 3x^4$ by $(x + 1)^2$.

29. $21a^5 - 21b^5$ by $7a - 7b$.

30. $64x^4 + 1$ by $8x^2 - 4x + 1$.

31. $50x + 9x^4 + 24 - 67x^2$ by $x + x^2 - 6$.

32. $x^4 + y^4 - 4xy^3 - 4x^3y + 6x^2y^2$ by $x^2 + y^2 - 2xy$.

33. $x^4 - 4x^3 + 2x^2 + 4x + 1$ by $(x - 1)^2 - 2$.

34. $9x^4 + 4y^4 - 37x^2y^2$ by $3x^2 - 2y^2 + 5xy$.

35. $a^4 + a^2b^2 + 25b^4$ by $(a - b)(a - 5b) + 3ab$.

36. $3x^2 + 4x + 6x^5 - 11x^3 - 4$ by $3x^2 - 4$.

37. $6x^5 + 15x^3 + 51x - 18$ by $2x^3 - 4x^2 + 7x - 2$.

38. $2x^4 - 11x - 4x^2 - 12 - 3x^3$ by $4 + 2x^2 + x$.

39. $m^5 - 48 - 17m^3 + 52m + 12m^2$ by $m - 2 + m^2$.

40. $x^{n+1} + x^ny - xy^n - y^{n+1}$ by $x^n - y^n$.

41. $x^5y - xy^5$ by $x^3 + y^3 + xy^2 + x^2y$.

42. $x^5 - 6x^2 - x - 6$ by $x^2 + 2x + 3$.

43. $2a^5 + 53a^2b^3 - 49b^5 - 7a^3b^2 - 9a^4b$ by $2a^2 - 5ab - 7b^2$.

44. $x^6 - 6x^4 + 5x^2 - 1$ by $x^3 + 2x^2 - x - 1$.

45. $2x^2 - 6y^2 - 12z^2 + xy - 2xz + 17yz$ by $2x + 4z - 3y$.

46. $a^{2n} - b^{2m} + 2b^m c^r - c^{2r}$ by $a^n + b^m - c^r$.

47. $x^6 - 1 - 6x^4 - 3x^2$ by $-2x^2 - x + x^3 - 1$.

48. $12a^5 - 14a^4b + 10a^3b^2 - a^2b^3 - 8ab^4 + 4b^5$
by $6a^3 - 4a^2b - 3ab^2 + 2b^3$.

The operation of division may be abridged in certain cases by the use of parentheses.

49. Divide $(a^2 + ab)x^2 + (2ac + bc + ad)x + c(c + d)$
by $ax + c$.

$$\begin{array}{l|l}
(a^2+ab)x^2 + (2ac + bc + ad)x + c(c+d) & ax + c \\ \cline{2-2}
\underline{(a^2+ab)x^2 + (\ \ ac + bc \qquad\ \ )x \qquad\qquad\quad} & (a+b)x + (c+d), \\
\qquad\qquad\quad (\ \ ac \qquad\quad + ad)x + c(c+d) & \qquad\qquad\quad \textit{Ans.} \\
\qquad\qquad\quad \underline{(\ \ ac \qquad\quad + ad)x + c(c+d)} &
\end{array}$$

Divide the following:

50. $x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$
by $x^2 + (b+c)x + bc$.

51. $(b + c)a^2 + (b^2 + 3bc + c^2)a + bc(b + c)$ by $a + b + c$.

52. $(x + y)^2 - 5(x + y) + 6$ by $(x + y) - 2$.

53. $(a + b)^3 + 1$ by $(a + b) + 1$.

54. $x^3 + (a + b - c)x^2 + (ab - bc - ca)x - abc$
by $x^2 + (b - c)x - bc$.

55. $(m - n)^4 - 2(m - n)^2 + 1$
by $(m - n)^2 - 2(m - n) + 1$.

56. $x^3 + (a - b + c)x^2 + (ac - ab - bc)x - abc$ by $x + c$.

57. $x^4 + (3 - b)x^3 + (c - 3b - 2)x^2 + (2b + 3c)x - 2c$
by $x^2 + 3x - 2$.

58. $a^2(b + c) + a(b^2 + bc + c^2) - bc(b + c)$ by $a + b + c$.

VII. FORMULÆ.

94. A **Formula** is the algebraic expression of a general rule.

95. The following results are of great importance in abridging algebraic operations:

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \quad\;\; ab+b^2 \\ \hline a^2+2ab+b^2 \end{array} \qquad \begin{array}{l} a-b \\ a-b \\ \hline a^2-ab \\ \quad\; -ab+b^2 \\ \hline a^2-2ab+b^2 \end{array} \qquad \begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad\; -ab-b^2 \\ \hline a^2 \qquad\;\; -b^2 \end{array}$$

In the first case, we have $(a+b)^2=a^2+2ab+b^2$. (1)

That is, *the square of the sum of two quantities is equal to the square of the first, plus twice the product of the two, plus the square of the second.*

In the second case, we have $(a-b)^2=a^2-2ab+b^2$. (2)

That is, *the square of the difference of two quantities is equal to the square of the first, minus twice the product of the two, plus the square of the second.*

In the third case, we have $(a+b)(a-b)=a^2-b^2$. (3)

That is, *the product of the sum and difference of two quantities is equal to the difference of their squares.*

EXAMPLES.

96. 1. Square $3a+2bc$.

The square of the first term is $9a^2$, twice the product of the terms is $12abc$, and the square of the second term is $4b^2c^2$. Hence, by formula (1),

$$(3a+2bc)^2=9a^2+12abc+4b^2c^2.$$

Note. The following rule for the square of a monomial is evident from the above:

Square the coefficient, and multiply the exponent of each letter by 2.

Thus, the square of $5a^2b$ is $25a^4b^2$.

2. Square $4x - 5$.

By formula (2), $(4x-5)^2 = 16x^2 - 40x + 25$, *Ans.*

3. Multiply $6a^2 + b$ by $6a^2 - b$.

By formula (3), $(6a^2+b)(6a^2-b) = 36a^4 - b^2$, *Ans.*

Write by inspection the values of the following:

4. $(x-4)^2$.

5. $(3+a)^2$.

6. $(x+3)(x-3)$.

7. $(3a+5)^2$.

8. $(2x+1)(2x-1)$.

9. $(7-2x)^2$.

10. $(2m+3n)^2$.

11. $(4ab-x)^2$.

12. $(5+7x)(5-7x)$.

13. $(x^4-y^2)^2$.

14. $(3x+11)(3x-11)$.

15. $(x^2y+4)^2$.

16. $(3x^3+13)^2$.

17. $(6a^2-b^2c)^2$.

18. $(5a+7b^2)(5a-7b^2)$.

19. $(13ab+5ac)^2$.

20. $(x^3+5x)(x^3-5x)$.

21. $(1-12xyz)^2$.

22. $(4x^2+3y^5)(4x^2-3y^5)$.

23. $(10x^3+9x^2)^2$.

24. $(4a^p-5b^q)^2$.

25. $(a^m+a^n)(a^m-a^n)$.

26. $(7x^3+11x)^2$.

27. $(5a^m-a^n)^2$.

28. Multiply $a+b+c$ by $a+b-c$.

$$\begin{aligned}(a+b+c)(a+b-c) &= [(a+b)+c][(a+b)-c] \\ &= (a+b)^2 - c^2, \quad \text{by formula (3)} \\ &= a^2+2ab+b^2-c^2, \text{ } \textit{Ans.}\end{aligned}$$

29. Multiply $a+b-c$ by $a-b+c$.

$$\begin{aligned}(a+b-c)(a-b+c)&=[a+(b-c)][a-(b-c)]\\&=a^2-(b-c)^2\\&=a^2-(b^2-2bc+c^2)\\&=a^2-b^2+2bc-c^2,\ \textit{Ans.}\end{aligned}$$

Expand the following:

30. $(x+y+z)(x-y+z)$.

31. $(x+y+z)(x-y-z)$.

32. $(1+a-b)(1-a+b)$.

33. $(x^2+x+1)(x^2-x-1)$.

34. $(a+b-c)(a-b-c)$.

35. $(a^2+2a+1)(a^2-2a+1)$.

36. $(x^2+2x-3)(x^2-2x+3)$.

37. $(m^2+mn+n^2)(m^2-mn+n^2)$.

97. We find by multiplication:

$$\begin{array}{l}x+5\\ \underline{x+3}\\ x^2+5x\\ \underline{\quad+3x+15}\\ x^2+8x+15\end{array}\qquad\qquad\begin{array}{l}x-5\\ \underline{x-3}\\ x^2-5x\\ \underline{\quad-3x+15}\\ x^2-8x+15\end{array}$$

$$\begin{array}{l}x+5\\ \underline{x-3}\\ x^2+5x\\ \underline{\quad-3x-15}\\ x^2+2x-15\end{array}\qquad\qquad\begin{array}{l}x-5\\ \underline{x+3}\\ x^2-5x\\ \underline{\quad+3x-15}\\ x^2-2x-15\end{array}$$

We observe in these products the following laws:

I. The coefficient of x is the algebraic sum of the numbers in the factors.

II. The last term is the product of the numbers.

By aid of the above laws the product of two binomials of the form $x+a$, $x+b$ may be written by inspection.

1. Required the value of $(x-8)(x+5)$.

The coefficient of x is -3; and the last term is -40.

Hence, $$(x-8)(x+5)=x^2-3x-40, \text{ } Ans.$$

EXAMPLES.

Write by inspection the values of the following:

2. $(x+7)(x+5)$.

3. $(x-3)(x-4)$.

4. $(x+8)(x-2)$.

5. $(x-3)(x+1)$.

6. $(x-5)(x+6)$.

7. $(x+1)(x+12)$.

8. $(x-7)(x+2)$.

9. $(x-8)(x-6)$.

10. $(x+9)(x-5)$.

11. $(x-8)(x-9)$.

12. $(x+4m)(x+6m)$.

13. $(x-5a)(x+a)$.

14. $(a+b)(a-4b)$.

15. $(a+5b)(a+8b)$.

16. $(x^2-3)(x^2-7)$.

17. $(x^3+2a)(x^3-6a)$.

98. The following results may be verified by division:

(1) $\dfrac{a^2-b^2}{a+b}=a-b.$

(2) $\dfrac{a^2-b^2}{a-b}=a+b.$

(3) $\dfrac{a^3+b^3}{a+b}=a^2-ab+b^2.$

(4) $\dfrac{a^3-b^3}{a-b}=a^2+ab+b^2.$

Formulæ (3) and (4) may be stated in words as follows:

If the sum of the cubes of two quantities be divided by the sum of the quantities, the quotient is equal to the square of the first quantity, minus the product of the two, plus the square of the second.

If the difference of the cubes of two quantities be divided by the difference of the quantities, the quotient is equal to the square of the first quantity, plus the product of the two, plus the square of the second.

EXAMPLES.

1. Divide $36y^2z^4 - 9$ by $6yz^2 + 3$.

By formula (1), $\dfrac{36y^2z^4 - 9}{6yz^2 + 3} = 6yz^2 - 3$, *Ans.*

2. Divide $1 + 8a^3$ by $1 + 2a$.

By formula (3), $\dfrac{1 + 8a^3}{1 + 2a} = 1 - 2a + 4a^2$, *Ans.*

3. Divide $27a^3 - b^3$ by $3a - b$.

By formula (4), $\dfrac{27a^3 - b^3}{3a - b} = 9a^2 + 3ab + b^2$, *Ans.*

EXAMPLES.

Write by inspection the values of the following:

4. $\dfrac{x^2 - 81}{x - 9}$.

5. $\dfrac{25 - 16a^2}{5 + 4a}$.

6. $\dfrac{x^3 + 1}{x + 1}$.

7. $\dfrac{1 - m^3}{1 - m}$.

8. $\dfrac{a^3 - 8}{a - 2}$.

9. $\dfrac{27 + x^3}{3 + x}$.

10. $\dfrac{x^6 - 16x^2}{x^3 + 4x}$.

11. $\dfrac{x^3 - 64}{x - 4}$.

12. $\dfrac{1 - 8m^3}{1 - 2m}$.

13. $\dfrac{a^3 + 343}{a + 7}$.

14. $\dfrac{x^6 - y^6}{x^2 - y^2}$.

15. $\dfrac{27 + x^3y^6}{3 + xy^2}$.

16. $\dfrac{49a^2 - 121b^4}{7a - 11b^2}$.

17. $\dfrac{64m^3 + n^6}{4m + n^2}$.

18. $\dfrac{x^3 + 125y^3}{x + 5y}$.

Divide the following:

19. $27x^3y^3 - 64z^3$ by $3xy - 4z$.

20. $25a^4 - 81b^2c^6$ by $5a^2 - 9bc^3$.

21. $343 + 125x^3y^3$ by $7 + 5xy$.

22. $64m^3 - 216n^6$ by $4m - 6n^2$.

23. $729x^9y^3 + 512z^6$ by $9x^3y + 8z^2$.

99. By actual division we obtain:

$$\frac{a^4 - b^4}{a + b} = a^3 - a^2b + ab^2 - b^3.$$

$$\frac{a^4 - b^4}{a - b} = a^3 + a^2b + ab^2 + b^3.$$

$$\frac{a^5 + b^5}{a + b} = a^4 - a^3b + a^2b^2 - ab^3 + b^4.$$

$$\frac{a^5 - b^5}{a - b} = a^4 + a^3b + a^2b^2 + ab^3 + b^4;\ \text{etc.}$$

In these results we observe the following laws:

I. The number of terms is the same as the exponent of a in the dividend.

II. The exponent of a in the first term is less by 1 than the exponent of a in the dividend, and decreases by 1 in each succeeding term.

III. The exponent of b in the second term is 1, and increases by 1 in each succeeding term.

IV. The terms are all positive when the divisor is $a - b$, and are alternately positive and negative when the divisor is $a + b$.

100. In connection with Art. 99, the following principles are of great importance:

If n is any whole number,

(1) *$a^n + b^n$ is divisible by $a + b$ if n is odd, and by neither $a + b$ nor $a - b$ if n is even.*

(2) *$a^n - b^n$ is divisible by $a - b$ if n is odd, and by both $a + b$ and $a - b$ if n is even.*

EXAMPLES.

101. 1. Divide $a^7 - b^7$ by $a - b$.

Applying the laws of Art. 99, we have

$$\frac{a^7 - b^7}{a - b} = a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6, \textit{ Ans.}$$

2. Divide $x^4 - 81$ by $x + 3$.

Since $81 = 3^4$, we have

$$\frac{x^4 - 3^4}{x + 3} = x^3 - 3x^2 + 3^2x - 3^3 = x^3 - 3x^2 + 9x - 27, \textit{ Ans.}$$

Write by inspection the values of the following:

3. $\frac{a^6 - b^6}{a - b}$.

4. $\frac{x^6 - y^6}{x + y}$.

5. $\frac{m^7 + n^7}{m + n}$.

6. $\frac{m^7 - n^7}{m - n}$.

7. $\frac{1 - x^4}{1 - x}$.

8. $\frac{x^4 - 16}{x - 2}$.

9. $\frac{1 - a^5}{1 - a}$.

10. $\frac{a^5 + 1}{a + 1}$.

11. $\frac{1 - n^6}{1 - n}$.

12. $\frac{x^4 - 81}{x - 3}$.

13. $\frac{x^5 - 32}{x - 2}$.

14. $\frac{a^6 - 64}{a + 2}$.

15. $\frac{a^{10} + b^{10}}{a^2 + b^2}$.

16. $\frac{x^7 - 128}{x - 2}$.

17. $\frac{x^5 + 243}{x + 3}$.

Divide the following:

18. $m^4 - 16n^8$ by $m - 2n^2$.

19. $x^8 - y^8z^8$ by $x - yz$.

20. $32a^5 + b^5$ by $2a + b$.

21. $m^5 - 243n^5$ by $m - 3n$.

22. $256x^4 - y^8$ by $4x + y^2$.

VIII. FACTORING.

102. **Factoring** is the process of resolving a quantity into its factors. (Art. 11.)

103. The factoring of monomials may be performed by inspection; thus,

$$12\,a^3b^2c = 2 \cdot 2 \cdot 3\,aaabbc.$$

A polynomial is not always factorable; but there are certain forms which can always be factored, the more important of which will be considered in the succeeding articles.

Case I.

104. *When the terms of the polynomial have a common monomial factor.*

1. Factor $a^3 + 3\,a$.

Each term contains the monomial factor a.

Dividing the expression by a, we have $a^2 + 3$. Hence,

$$a^3 + 3\,a = a(a^2 + 3), \text{ } Ans.$$

2. Factor $14\,xy^4 - 35\,x^3y^2$.

$$14xy^4 - 35x^3y^2 = 7\,xy^2(2y^2 - 5x^2), \text{ } Ans.$$

EXAMPLES.

Factor the following:

3. $x^2 + 5\,x$.

4. $3\,m^3 - 12\,m^2$.

5. $16\,a^4 - 12\,a$.

6. $27\,c^4d^2 + 9\,c^3d$.

7. $60\,m^2n^4 - 12\,m^3$.

8. $5\,x^3 + 10\,x^2 + 15\,x$.

9. $a^5 - 2\,a^4 + 3\,a^3 - a^2$.

10. $36\,x^3y - 60\,x^2y^4 - 84\,x^4y^2$.

11. $21\,m^3n + 35\,mn^3 - 14\,mn$.

12. $84\,x^2y^3 - 140\,x^3y^4 + 70\,x^4y^5$.

13. Factor the sum of $54a^4b^3$, $-72a^3c^2$, and $-90a^2d$.

14. Factor the sum of $96c^4d^5$, $120c^2d^7$, and $-144c^5d^4$.

Case II.

105. *When the polynomial consists of four terms, of which the first two and the last two have a common binomial factor.*

1. Factor $am - bm + an - bn$.

Factoring the first two and last two terms as in Case I, we have

$$m(a - b) + n(a - b).$$

Each term now contains the binomial factor $a - b$. Dividing the expression by $a - b$, we obtain $m + n$. Hence,

$$am - bm + an - bn = (a - b)(m + n), \ Ans.$$

2. Factor $am - bm - an + bn$.

$$\begin{aligned} am - bm - an + bn &= am - bm - (an - bn) \\ &= m(a - b) - n(a - b) \\ &= (a - b)(m - n), \ Ans. \end{aligned}$$

Note. If the third term is negative, as in Ex. 2, it is convenient, before factoring, to enclose the last two terms in a parenthesis preceded by a $-$ sign.

EXAMPLES.

Factor the following:

3. $ab + bx + ay + xy$.
4. $ac - cm + ad - dm$.
5. $x^2 + 2x - xy - 2y$.
6. $x^2 - ax - bx + ab$.
7. $a^3 - a^2b + ab^2 - b^3$.
8. $a^3 - a^2b - ab^2 + b^3$.
9. $x^2 + ax - bx - ab$.
10. $mx^2 - my^2 + nx^2 - ny^2$.
11. $x^3 + x^2 + x + 1$.
12. $6x^3 + 4x^2 - 9x - 6$.
13. $8cx - 12cy + 2dx - 3dy$.
14. $6n - 21m^2n - 8m + 28m^3$.

106. If a quantity can be resolved into two equal factors, it is said to be a *perfect square*, and one of the equal factors is called its square root.

Thus, since $9a^4b^2$ equals $3a^2b \times 3a^2b$, it is a perfect square, and $3a^2b$ is its square root.

Note. $9a^4b^2$ also equals $-3a^2b \times -3a^2b$, so that its square root is either $3a^2b$ or $-3a^2b$. In the examples in this chapter we shall consider the *positive* square root only.

107. The following rule for extracting the square root of a monomial is evident from Art. 106:

Extract the square root of the coefficient, and divide the exponent of each letter by 2.

For example, the square root of $25x^6y^8z^2$ is $5x^3y^4z$.

108. It follows from Art. 95 that a trinomial is a perfect square when its first and last terms are perfect squares and positive, and the second term is twice the product of their square roots.

Thus, $4x^2 - 12xy + 9y^2$ is a perfect square.

109. To find the square root of a perfect trinomial square, we take the converse of the rules of Art. 95:

Extract the square roots of the first and last terms, and connect the results by the sign of the second term.

Thus, let it be required to find the square root of

$$4x^2 - 12xy + 9y^2.$$

The square root of the first term is $2x$, and of the last term $3y$; and the sign of the second term is —. Hence the required square root is

$$2x - 3y.$$

Case III.

110. *When a trinomial is a perfect square* (Art. 108).

1. Factor $a^2 + 2ab^2 + b^4$.

By Art. 109, the square root of the expression is $a + b^2$. Hence,

$$a^2 + 2ab^2 + b^4 = (a + b^2)(a + b^2), \text{ or } (a + b^2)^2, \textit{ Ans.}$$

2. Factor $4x^2 - 12xy + 9y^2$.

$$4x^2 - 12xy + 9y^2 = (2x - 3y)(2x - 3y)$$
$$= (2x - 3y)^2, \textit{ Ans.}$$

Note. The given expression may be written $9y^2 - 12xy + 4x^2$; whence,

$$9y^2 - 12xy + 4x^2 = (3y - 2x)(3y - 2x) = (3y - 2x)^2;$$

which is another form of the answer.

EXAMPLES.

Factor the following:

3. $x^2 + 2xy + y^2$.
4. $4 + 4m + m^2$.
5. $x^2 - 14x + 49$.
6. $a^2 - 10a + 25$.
7. $y^2 + 2y + 1$.
8. $m^2 - 2m + 1$.
9. $x^4 + 12x^2 + 36$.
10. $n^6 - 20n^3 + 100$.
11. $x^2y^2 + 16xy + 64$.
12. $1 - 10ab + 25a^2b^2$.
13. $16m^2 - 8am + a^2$.
14. $a^4 + 2a^3 + a^2$.
15. $x^6 - 4x^4 + 4x^2$.
16. $36m^2 - 36mn + 9n^2$.
17. $4a^2 + 44ab + 121b^2$.
18. $x^6 + 8x^5 + 16x^4$.
19. $a^2b^4 + 18ab^2c + 81c^2$.
20. $25x^2 - 70xyz + 49y^2z^2$.
21. $9x^8 - 66x^6 + 121x^4$.
22. $9a^4 + 60a^2bc^2d + 100b^2c^4d^2$.
23. $64x^8 - 160x^7 + 100x^6$.
24. $4a^4b^2 + 52a^3b^3 + 169a^2b^4$.
25. $16x^4 - 120mnx^2 + 225m^2n^2$.
26. $(a - b)^2 + 2(a - b) + 1$.
27. $(x + y)^2 - 16(x + y) + 64$.
28. $(x^2 - x)^2 + 6(x^2 - x) + 9$.

Case IV.

111. *When an expression is the difference of two perfect squares.*

Comparing with the third case of Art. 95, we see that such an expression is the product of the sum and difference of two quantities.

Therefore, to obtain the factors, we take the converse of the rule of Art. 95:

Extract the square root of the first term and of the last term; add the results for one factor, and subtract the second result from the first for the other.

1. Factor $36x^2 - 49y^2$.

The square root of the first term is $6x$, and of the last term $7y$. Hence, by the rule,

$$36x^2 - 49y^2 = (6x + 7y)(6x - 7y), \quad \textit{Ans.}$$

2. Factor $(2x - 3y)^2 - (x - y)^2$.

$$\begin{aligned}(2x - 3y)^2 - (x - y)^2 \\ &= [(2x - 3y) + (x - y)][(2x - 3y) - (x - y)] \\ &= (2x - 3y + x - y)(2x - 3y - x + y) \\ &= (3x - 4y)(x - 2y), \quad \textit{Ans.}\end{aligned}$$

EXAMPLES.

Factor the following:

3. $x^2 - y^2$.

4. $x^2 - 1$.

5. $4 - a^2$.

6. $9m^2 - 4$.

7. $9x^2 - 16y^2$.

8. $25a^2 - b^4$.

9. $1 - 49x^2y^2$.

10. $a^2b^6 - c^4d^8$.

11. $49m^2 - 100n^6$.

12. $36x^4 - 81y^2$.

13. $64a^2 - 121b^2c^2$.

14. $144x^2y^4 - 225z^6$.

15. $(a+b)^2-(c+d)^2$.

16. $(a-c)^2-b^2$.

17. $m^2-(x-y)^2$.

18. $m^4-(m-1)^2$.

19. $(x-c)^2-(y-d)^2$.

20. $(a-3)^2-(b+2)^2$.

21. $(2x+m)^2-(x-m)^2$.

22. $(3a+5)^2-(2a-3)^2$.

It is sometimes possible to express a polynomial in the form of the difference of two perfect squares, when it may be factored by the rule of Case IV.

23. Factor $2mn+m^2-1+n^2$.

The expression may be written $m^2+2mn+n^2-1$, which, by Case III., is equivalent to $(m+n)^2-1$. Hence, by the rule,

$$(m+n)^2-1=(m+n+1)(m+n-1), \text{ Ans.}$$

24. Factor $2xy+1-x^2-y^2$.

$$2xy+1-x^2-y^2=1-x^2+2xy-y^2=1-(x^2-2xy+y^2).$$

By Case III., this may be written $1-(x-y)^2$. Hence the factors are

$$[1+(x-y)][1-(x-y)]=(1+x-y)(1-x+y), \text{ Ans.}$$

25. Factor $2xy+b^2-x^2-2ab-y^2+a^2$.

$$\begin{aligned}2xy+b^2-x^2-2ab-y^2+a^2 & \\ &=a^2-2ab+b^2-x^2+2xy-y^2\\ &=a^2-2ab+b^2-(x^2-2xy+y^2)\\ &=(a-b)^2-(x-y)^2, \text{ by Case III.}\\ &=[(a-b)+(x-y)][(a-b)-(x-y)]\\ &=(a-b+x-y)(a-b-x+y), \text{ Ans.}\end{aligned}$$

Factor the following:

26. $x^2+2xy+y^2-4$.

27. $a^2-2ab+b^2-c^2$.

28. $a^2-b^2+2bc-c^2$.

29. $a^2-b^2-2bc-c^2$.

30. $c^2 - 1 + d^2 + 2cd.$

31. $9 - x^2 - y^2 + 2xy.$

32. $4b - 1 - 4b^2 + 4m^4.$

33. $4a^2 + b^2 - 9d^2 - 4ab.$

34. $a^2 - 2am + m^2 - b^2 - 2bn - n^2.$

35. $x^2 - y^2 + c^2 - d^2 - 2cx + 2dy.$

36. $a^2 - b^2 + m^2 - n^2 + 2am + 2bn.$

37. $a^2 - b^2 + c^2 - d^2 + 2ac - 2bd.$

Case V.

112. *When an expression is a trinomial of the form* $x^2 + ax + b$.

In Art. 97 we derived a rule for the product of two binomials of the form $x + a$, $x + b$, by considering the following cases in multiplication:

1. $(x + 5)(x + 3) = x^2 + 8x + 15.$
2. $(x - 5)(x - 3) = x^2 - 8x + 15.$
3. $(x + 5)(x - 3) = x^2 + 2x - 15.$
4. $(x - 5)(x + 3) = x^2 - 2x - 15.$

In certain cases it is possible to reverse the operation, and resolve a trinomial of the form $x^2 + ax + b$ into the product of two binomial factors.

The first term of each factor will obviously be x; and to obtain the second terms, we take the converse of the rule of Art. 97:

Find two numbers whose algebraic sum is the coefficient of x, *and whose product is the last term.*

Thus, let it be required to factor $x^2 - 5x - 24$.

The coefficient of x is -5, and the last term is -24; we are then to find two numbers whose algebraic sum is -5, and product -24. By inspection we determine that the numbers are -8 and 3. Hence,

$$x^2 - 5x - 24 = (x - 8)(x + 3).$$

113. The work of finding the numbers may be abridged by the following considerations:

1. When the last term of the product is $+$, as in Exs. 1 and 2, the coefficient of x is the *sum* of the numbers; both numbers being $+$ when the second term is $+$, and $-$ when the second term is $-$.

2. When the last term of the product is $-$, as in Exs. 3 and 4, the coefficient of x is the *difference* of the numbers (disregarding signs); the greater number having the same sign as the second term, and the smaller number the opposite sign.

We may embody these observations in two rules, which will be found more convenient than the rule of Art. 112 in the solution of examples:

I. *If the last term is $+$, find two numbers whose sum is the coefficient of x, and whose product is the last term; and give to both numbers the sign of the second term.*

II. *If the last term is $-$, find two numbers whose difference is the coefficient of x, and whose product is the last term; give to the greater number the sign of the second term, and to the smaller number the opposite sign.*

Note. By the expressions "coefficient of x" and "last term," in the above rules, we understand their *absolute values,* without regard to sign.

EXAMPLES.

114. 1. Factor $x^2+14x+45$.

According to Rule I., we find two numbers whose sum is 14, and product 45. The numbers are 9 and 5; and as the second term is $+$, both numbers are $+$. Hence,

$$x^2+14x+45=(x+9)(x+5),\ \textit{Ans.}$$

2. Factor x^2-6x+5.

By Rule I., we find two numbers whose sum is 6, and

product 5. The numbers are 5 and 1; and as the second term is $-$, both numbers are $-$. Hence,

$$x^2 - 6x + 5 = (x-5)(x-1), \text{ Ans.}$$

3. Factor $x^2 + 5x - 14$.

By Rule II., we find two numbers whose difference is 5, and product 14. The numbers are 7 and 2; and as the second term is $+$, the greater number is $+$, and the smaller number $-$. Hence,

$$x^2 + 5x - 14 = (x+7)(x-2), \text{ Ans.}$$

4. Factor $x^2 - 5x - 24$.

By Rule II., we find two numbers whose difference is 5, and product 24. The numbers are 8 and 3; and as the second term is $-$, the greater number is $-$, and the smaller number $+$. Hence,

$$x^2 - 5x - 24 = (x-8)(x+3), \text{ Ans.}$$

Factor the following:

5. $x^2 + 5x + 6$.

6. $x^2 - 3x + 2$.

7. $y^2 + 2y - 8$.

8. $m^2 - 7m - 30$.

9. $a^2 - 11a + 18$.

10. $x^2 + x - 6$.

11. $c^2 + 9c + 8$.

12. $y^2 - 2y - 35$.

13. $a^2 + 13a - 48$.

14. $x^2 - 10x + 21$.

15. $x^2 + 13x + 36$.

16. $n^2 - n - 90$.

17. $x^2 - 6x - 16$.

18. $m^2 + 16m + 63$.

19. $a^2 - 15a + 44$.

20. $y^2 + 7y - 60$.

21. $x^2 - 11x + 10$.

22. $m^2 + 2m - 80$.

23. $n^2 + 23n + 102$.

24. $x^2 - 9x - 90$.

25. $a^2 - 11a - 26$.

26. $x^2 + x - 42$.

27. $c^2 - 18c + 32$.

28. $m^2 - 8m - 33$.

29. $x^2+20x+75.$

30. $x^2+4x-96.$

31. $y^2-17y-110.$

32. $x^2-19x+78.$

33. $x^2+7x-98.$

34. $a^2+22a+105.$

35. $x^2-23x+130.$

36. $a^4+10a^2-144.$

37. $x^4-19x^2-120.$

38. $c^6+12c^3+11.$

39. $x^2y^6+2xy^3-120.$

40. $a^2b^4-7ab^2-144.$

41. $n^2x^2+25nx+100.$

42. $y^8-20y^4+91.$

43. $a^4b^4-2a^2b^2-48.$

44. $m^4+26m^2-87.$

45. Factor $x^4+5abx^2-84a^2b^2$.

We find two numbers whose difference is 5, and product 84. The numbers are 12 and 7; and, by the rule, the greater is $+$, and the smaller $-$. Hence,

$$x^4+5abx^2-84a^2b^2=(x^2+12ab)(x^2-7ab), \textit{ Ans.}$$

46. Factor $1-6a-27a^2$.

The numbers whose difference is 6, and product 27, are 9 and 3. Hence,

$$1-6a-27a^2=(1-9a)(1+3a), \textit{ Ans.}$$

Factor the following:

47. $a^2-3ax+2x^2.$

48. $x^2+5xy-66y^2.$

49. $1+13a+42a^2.$

50. $m^2-15mn+56n^2.$

51. $a^2-ab-56b^2.$

52. $a^2b^2+4abc-45c^2.$

53. $1-3x-10x^2.$

54. $a^4+15a^3+44a^2.$

55. $z^2-10xy^2z-39x^2y^4.$

56. $(a+b)^2+5(a+b)+4.$

57. $1-9a+8a^2.$

58. $b^4+9ab^2-52a^2.$

59. $(m-n)^2+(m-n)-2.$

60. $x^6-5x^4-50x^2.$

61. $a^2+8ab+12b^2.$

62. $1-13xy+40x^2y^2.$

63. $(a-b)^2-3(a-b)-4.$

64. $x^4y^4+8x^2y^2z-48z^2.$

115. If a quantity can be resolved into three equal factors, it is said to be a *perfect cube*, and one of the equal factors is called its *cube root.*

Thus, since $27a^6b^3$ equals $3a^2b \times 3a^2b \times 3a^2b$, it is a perfect cube, and $3a^2b$ is its cube root.

116. It is evident from the above that the cube root of a monomial may be found by *extracting the cube root of the coefficient and dividing the exponent of each letter by* 3.

Thus, the cube root of $125x^6y^9z^3$ is $5x^2y^3z$.

Case VI.

117. *When an expression is the sum or difference of two perfect cubes.*

By Art. 98, the sum or difference of two perfect cubes is divisible by the sum or difference of their cube roots; and in either case, the quotient may be written by inspection by aid of the rules of Art. 98.

EXAMPLES.

1. Factor a^3+1.

The cube root of a^3 is a, and of 1 is 1; hence, one factor is $a+1$.

Dividing the expression by $a+1$, we have the quotient a^2-a+1 (Art. 98). Hence,

$$a^3+1=(a+1)(a^2-a+1),\ \textit{Ans.}$$

2. Factor $27x^3-64y^3$.

The cube root of $27x^3$ is $3x$, and of $64y^3$ is $4y$; hence, one factor is $3x-4y$. By Art. 98, the other factor is $9x^2+12xy+16y^2$. Hence,

$$27x^3-64y^3=(3x-4y)(9x^2+12xy+16y^2),\ \textit{Ans.}$$

Factor the following:

3. $a^3 + x^3$.
4. $m^3 - n^3$.
5. $x^3 - 1$.
6. $a^3b^3 + c^3$.
7. $1 - 8x^3$.
8. $a^6 + b^6$.
9. $x^6 + 1$.
10. $27x^3 - 1$.
11. $8c^6 - d^9$.
12. $27 + 8a^3$.
13. $m^3 - 64n^6$.
14. $64x^3 - 125$.
15. $125a^3 + 27m^3$.
16. $64c^3d^9 + 27$.
17. $125 - 8a^3b^6$.

Case VII.

118. *When an expression is the sum or difference of two equal odd powers of two quantities.*

By Art. 100, the sum or difference of two equal odd powers is divisible by the sum or difference of the quantities; and in either case, the quotient may be written by inspection by aid of the laws of Art. 99.

EXAMPLES.

1. Factor $a^5 + b^5$.

By Art. 100, one factor is $a + b$. Dividing the expression by $a + b$, the quotient is $a^4 - a^3b + a^2b^2 - ab^3 + b^4$ (Art. 99). Hence,

$$a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4), \textit{ Ans.}$$

Factor the following:

2. $a^5 - b^5$.
3. $x^5 + 1$.
4. $1 - a^5$.
5. $m^7 + n^7$.
6. $x^7 - y^7$.
7. $a^7 - 1$.
8. $c^5 - m^5n^5$.
9. $1 + 32n^5$.
10. $243x^5 - y^5$.
11. $x^7 + 128$.
12. $32 - 243a^5$.

119. By applying one or more of the rules already given, an expression may often be separated into more than two factors.

1. Factor $2ax^3y^2 - 8axy^4$.

By Case I., $2ax^3y^2 - 8axy^4 = 2axy^2(x^2 - 4y^2)$.

Factoring the quantity in the parenthesis by Case IV.,

$$2ax^3y^2 - 8axy^4 = 2axy^2(x + 2y)(x - 2y), \text{ } Ans.$$

2. Factor $m^6 - n^6$.

By Case IV., $m^6 - n^6 = (m^3 + n^3)(m^3 - n^3)$.

By Case VI., $m^3 + n^3 = (m + n)(m^2 - mn + n^2)$,

and $m^3 - n^3 = (m - n)(m^2 + mn + n^2)$.

Hence,

$$m^6 - n^6 = (m + n)(m - n)(m^2 - mn + n^2)(m^2 + mn + n^2), \text{ } Ans.$$

3. Factor $x^8 - y^8$.

By Case IV.,

$$\begin{aligned} x^8 - y^8 &= (x^4 + y^4)(x^4 - y^4) \\ &= (x^4 + y^4)(x^2 + y^2)(x^2 - y^2) \\ &= (x^4 + y^4)(x^2 + y^2)(x + y)(x - y), \text{ } Ans. \end{aligned}$$

MISCELLANEOUS EXAMPLES.

120. In factoring the following expressions, the common monomial factors should be first removed, as shown in Example 1 of the preceding article.

1. $6a^2x^3 - 6a^4x$.
2. $1 - 4x + 4x^2$.
3. $x^6 - 1$.
4. $a^2 + 9a + 18$.
5. $x^2 + ax + bx + ab$.
6. $m^2 - 7m - 8$.
7. $2x^5 + x$.
8. $5a^3 - 5$.
9. $a^5b^5 - c^5d^5$.
10. $x^4 - 16$.
11. $a^5 - a^4 + a^3 - a^2$.
12. $3x^2 + 27x + 42$.
13. $x^2 - (2y - 3z)^2$.
14. $a^2 + 20ab + 100b^2$.

15. $5a^2bc - 10ab^2c - 15abc^2$.
16. $3a^4 - 21a^3 + 30a^2$.
17. $x^3 + 8y^3z^3$.
18. $2a^5 - 2a$.
19. $1 - a^2 - b^2 + 2ab$.
20. $x^2 - 8x + 7$.
21. $1 + 12x + 27x^2$.
22. $18x^3y - 2xy^3$.
23. $x^9 - x^2$.
24. $4x^2y^4 + 28xy^2 + 49$.
25. $a^6 + 6a^3 - 40$.
26. $a^2 - 18ab - 40b^2$.
27. $2x^3y + 2xy^3 - 2xyz^2 + 4x^2y^2$.
28. $12m^3n - 18m^2n^2 + 24mn^3$.
29. $32a^4b + 4ab^4$.
30. $x^4 - 81$.
31. $y - y^9$.
32. $x^3 + 2x^2 - x - 2$.
33. $x^2 + 7x^3 - 30x^4$.
34. $(3x + y)^2 - (x - 2y)^2$.
35. $m^2x^6 - 8mx^3 - 65$.
36. $135x^5 - 5x^2$.
37. $2x^3y - 2x^2y^2 - 60xy^3$.
38. $80x^2y^5 - 5x^6y$.
39. $3a^3b + 18a^2b + 27ab$.
40. $(x^2 + y^2 - z^2)^2 - 4x^2y^2$.
41. $a^2bc - ac^2d - ab^2d + bcd^2$.
42. $a^2 - 14ab + 33b^2$.
43. $3x^6y + 3xy^6$.
44. $4m^4 - 20m^2n + 25n^2$.
45. $3a^3b + 3a^2b^2 - 6ab^3$.
46. $a^2x^2 - a^2y^2 - b^2x^2 + b^2y^2$.
47. $(a - 2b)^2 - 2(a - 2b) - 8$.
48. $100x^2y^4 - 81z^2$.
49. $a^6 - 64$.
50. $x^4 - (x - 6)^2$.
51. $(a^2 + 3a)^2 - 14(a^2 + 3a) + 40$.
52. $(4m + n)^2 - (2m - 3n)^2$.
53. $(a^2 - b^2 - c^2)^2 - 4b^2c^2$.
54. $1000 + 27m^6$.
55. $x^3 - x^2 - x + 1$.
56. $3(a^2 - b^2) - (a - b)^2$.
57. $a^2 + b^2 - c^2 - d^2 - 2ab - 2cd$.
58. $(x^2 + 4)^2 - 16x^2$.
59. $x^3 - y^3 - 3xy(x - y)$.
60. $(a^2 + a - 4)^2 - 4$.

IX. HIGHEST COMMON FACTOR.

121. A **Common Factor** of two or more quantities is a quantity which will divide each of them without a remainder.

Thus, $2xy^2$ is a common factor of $12x^3y^3$ and $20x^2y^4$.

122. A *prime quantity* is one which cannot be divided, without a remainder, by any integral quantity except itself or unity.

For example, a, b, and $a+c$ are prime quantities.

123. Two quantities are said to be *prime to each other* when they have no common factor except unity.

Thus, $2a$ and $3b^2$ are prime to each other.

124. The **Highest Common Factor** of two or more quantities is the product of all the prime factors common to those quantities.

It is evident from this definition that the highest common factor of two or more quantities is the expression of *highest degree* (Art. 33) which will divide each of them without a remainder.

Thus, the highest common factor of x^3y^3 and x^2y^4 is x^2y^3.

125. In determining the highest common factor of algebraic quantities, it is convenient to distinguish three cases.

Case I.

126. *When the quantities are monomials.*

1. Find the H.C.F. of $42a^3b^2$, $70a^2bc$, and $98a^4b^3d^2$.

$$\begin{aligned} 42a^3b^2 &= 2\cdot 3\cdot 7\cdot a^3b^2 \\ 70a^2bc &= 2\cdot 5\cdot 7\cdot a^2bc \\ 98a^4b^3d^2 &= 2\cdot 7\cdot 7\cdot a^4b^3d^2 \end{aligned}$$

Hence, the H.C.F. $= 2\cdot 7\cdot a^2b$ (Art. 124) $= 14a^2b$, *Ans.*

RULE.

To the highest common factor of the coefficients, annex the common letters, giving to each the lowest exponent with which it occurs in any of the given quantities.

EXAMPLES.

Find the highest common factors of the following:

2. a^3x^2, $7a^4x$.

3. $15cd^2$, $9c^2d$.

4. $54a^3b$, $90ac^2$.

5. $18mn^5$, $45m^2n$, $72m^3n^2$.

6. $112xy^3z^2$, $154x^2yz^3$.

7. $15a^2x$, $45a^3y^2$, $60a^4z^3$.

8. $108x^3y^2z^7$, $144xy^3z^4$, $120x^5y^4z^5$.

9. $96a^5b^4$, $120a^3b^5$, $168a^4b^6$

10. $51a^2m^4n$, $85a^3m^3x$, $119a^4m^2y^4$.

CASE II.

127. *When the quantities are polynomials which can be readily factored by inspection.*

EXAMPLES.

1. Find the H.C.F. of

$$5x^3y - 15x^2y \text{ and } 10x^3y + 40x^2y - 210xy.$$

By the methods of Chapter VIII.,

$$5x^3y - 15x^2y = 5x^2y(x-3)$$

$$\begin{aligned} 10x^3y + 40x^2y - 210xy &= 10xy(x^2+4x-21) \\ &= 10xy(x+7)(x-3). \end{aligned}$$

In this case the common factors are 5, x, y, and $x-3$. Hence, the H.C.F. $= 5xy(x-3)$, *Ans.*

2. Find the H.C.F. of

$$4x^2 - 4x + 1,\ 4x^2 - 1, \text{ and } 2ax - a - 2bx + b.$$

$$4x^2 - 4x + 1 = (2x-1)(2x-1)$$
$$4x^2 - 1 = (2x+1)(2x-1)$$
$$2ax - a - 2bx + b = \underline{(a-b)(2x-1)}$$

Hence, the H.C.F. $= 2x - 1$, *Ans.*

Find the highest common factors of the following:

3. $3ax^2 - 2a^2x$ and $a^2x^2 - 3abx$.

4. $x^2 - y^2$ and $x^3 + y^3$.

5. $9a^4 - 4b^2$ and $(3a^2 - 2b)^2$.

6. $2x^6 - 2x^2$ and $6x^3 - 6x$.

7. $3cx + 21c - 3dx - 21d$ and $x^2 - 3x - 70$.

8. $m^3n + 2m^2n^2 + mn^3$ and $m^4n + mn^4$.

9. $3x^3 + 9x^2 - 120x$ and $3ax^2 - 9ax - 30a$.

10. $3xy - 4y + 3xz - 4z$ and $9x^2 - 16$.

11. $x^2 - x - 42$, $x^2 - 4x - 60$, and $x^2 + 12x + 36$.

12. $a^2 - 1$, $a^3 + 1$, and $a^2 + 2a + 1$.

13. $4x^2 - 12x + 9$, $4x^2 - 9$, and $4m^2nx - 6m^2n$.

14. $x^3 - x$, $x^3 + 9x^2 - 10x$, and $x^6 - x$.

15. $a^3 - 8b^3$, $a^2 - ab - 2b^2$, and $a^2 - 4ab + 4b^2$.

16. $2x^3 + 2x^2 - 4x$, $3x^4 + 6x^3 - 9x^2$, and $4x^5 - 20x^4 + 16x^3$.

17. $8m^3 - 125$, $4m^2 - 25$, and $4m^2 - 20m + 25$.

18. $x^4 - 16$, $x^2 - x - 6$, and $(x^2 - 4)^2$.

19. $3ax^6 - 3ax^5$, $ax^3 - 9ax^2 + 8ax$, and $2ax^5 - 2ax$.

20. $a^2 - b^2$, $ab - b^2 + ac - bc$, and $a^3 - a^2b + ab^2 - b^3$.

21. $12ax - 3a + 8cx - 2c$, $16x^2 - 1$, and $16x^2 - 8x + 1$.

Case III.

128. *When the quantities are polynomials which cannot be readily factored by inspection.*

The rule in Arithmetic for the H.C.F. of two numbers, is

Divide the greater number by the less; if there is a remainder, divide the divisor by it; and so on; continuing the operation until there is no remainder. Then the last divisor is the highest common factor required.

For example, required the H.C.F. of 169 and 546.

$$\begin{array}{l} 169)546(3 \\ \quad\;\; \underline{507} \\ \quad\;\; 39)169(4 \\ \qquad\quad\; \underline{156} \\ \qquad\quad\; 13)39(3 \\ \qquad\qquad\;\; \underline{39} \end{array}$$

Therefore 13 is the H.C.F. required.

129. We will now prove that a similar rule holds for the H.C.F. of two algebraic quantities.

Let A and B be two expressions, the degree of A being not lower than that of B. Suppose that B is contained in A p times with a remainder C; that C is contained in B q times with a remainder D; and that D is contained in C r times with no remainder. To prove that D is the H.C.F. of A and B.

The operation of division is shown as follows:

$$\begin{array}{l} B)\ A\ (p \\ \quad\; \underline{pB} \\ \quad\; C)\ B\ (q \\ \qquad\quad \underline{qC} \\ \qquad\quad D)\ C\ (r \\ \qquad\qquad\;\; \underline{rD} \end{array}$$

We will first prove that D is a common factor of A and B.

From the nature of subtraction, the minuend is equal to the sum of the subtrahend and remainder (Art. 59).

Hence,
$$A = pB + C \qquad (1)$$
$$B = qC + D \qquad (2)$$
$$C = rD$$

Substituting the value of C in (2), we have
$$B = qrD + D = D(qr + 1) \qquad (3)$$

Substituting the values of B and C in (1), we have
$$A = pD(qr + 1) + rD = D(pqr + p + r) \qquad (4)$$

From (3) and (4) we see that D is a common factor of A and B.

We will next prove that every common factor of A and B is a factor of D.

Let K be any common factor of A and B, such that $A = mK$, and $B = nK$. From the operation of division, we see that
$$C = A - pB \qquad (5)$$
$$D = B - qC \qquad (6)$$

Substituting the values of A and B in (5), we have
$$C = mK - pnK.$$

Substituting the values of B and C in (6), we have
$$D = nK - q(mK - pnK) = K(n - qm + pqn).$$

Hence K is a factor of D.

Therefore, since every common factor of A and B is a factor of D, and since D is itself a common factor of A and B, it follows that D is the highest common factor of A and B.

130. Hence, to find the H.C.F. of two algebraic expressions, A and B, of which the degree of A is not lower than that of B,

Divide A by B; if there is a remainder, divide the divisor by it; and continue thus to make the remainder the divisor, and the preceding divisor the dividend, until there is no remainder. Then the last divisor is the highest common factor required.

Note 1. Each division should be continued until the remainder is of a lower degree than the divisor.

Note 2. It is important to keep the work in the same order of powers of some common letter, as in ordinary division.

1. Find the H.C.F. of

$$18x^3 - 51x^2 + 13x + 5 \text{ and } 6x^2 - 13x - 5.$$

$$\begin{array}{rl}
6x^2-13x-5)\,18x^3-51x^2+13x+\ 5\,(3x-2 & \\
\underline{18x^3-39x^2-15x\quad\ } & \\
-12x^2+28x+\ 5 & \\
\underline{-12x^2+26x+10} & \\
2x-\ 5 &)\,6x^2-13x-5\,(3x+1 \\
& \underline{6x^2-15x\quad\ } \\
& \quad 2x-5 \\
& \quad \underline{2x-5}
\end{array}$$

Hence, $2x - 5$ is the H.C.F. required.

Note 3. Either of the given expressions may be divided by any quantity which is not a factor of the other, as such a quantity can evidently form no part of the highest common factor. Similarly, any remainder may be divided by a quantity which is not a common factor of the given expressions.

2. Find the H.C.F. of

$$6x^3 - 25x^2 + 14x \text{ and } 6ax^2 + 11ax - 10a.$$

Dividing the first expression by x, and the second by a, we have

$$\begin{array}{r}
6x^2 - 25x + 14)\,6x^2 + 11x - 10\,(1 \\
\underline{6x^2 - 25x + 14} \\
36x - 24
\end{array}$$

Dividing the remainder by 12,

$$
\begin{array}{r|l}
3x-2)\,6x^2-25x+14\,(2x-7 & \\
\underline{6x^2-4x} & \\
-21x+14 & \\
\underline{-21x+14} &
\end{array}
$$

Hence, $3x-2$ is the H.C.F. required.

Note 4. If the first term of a remainder is negative, the sign of each term may be changed.

3. Find the H.C.F. of $2x^2-3x-2$ and $2x^2-5x-3$.

$$
\begin{array}{r}
2x^2-3x-2)\,2x^2-5x-3\,(1 \\
\underline{2x^2-3x-2} \\
-2x-1
\end{array}
$$

Changing the sign of each term of this remainder,

$$
\begin{array}{r}
2x+1)\,2x^2-3x-2\,(x-2 \\
\underline{2x^2+x} \\
-4x-2 \\
\underline{-4x-2}
\end{array}
$$

Hence, $2x+1$ is the H.C.F. required.

Note 5. If the first term of the dividend or of any remainder is not divisible by the first term of the divisor, it may be made so by multiplying the dividend or remainder by any quantity which is not a factor of the divisor.

4. Find the H.C.F. of

$$2x^3-7x^2+5x-6 \text{ and } 3x^3-7x^2-7x+3.$$

Since $3x^3$ is not divisible by $2x^3$, we multiply the second quantity by 2.

$$
\begin{array}{r}
2x^3-7x^2+5x-6)\,6x^3-14x^2-14x+6\,(3 \\
\underline{6x^3-21x^2+15x-18} \\
7x^2-29x+24
\end{array}
$$

Since $2x^3$ is not divisible by $7x^2$, we multiply each term of the new dividend by 7.

$$\begin{array}{r}7x^2-29x+24)14x^3-49x^2+35x-42(2x\\ \underline{14x^3-58x^2+48x}\\ 9x^2-13x-42\end{array}$$

Multiplying this by 7 to make its first term divisible by $7x^2$,

$$\begin{array}{r}7x^2-29x+24)63x^2-91x-294(9\\ \underline{63x^2-261x+216}\\ 170x-510\end{array}$$

Dividing by 170,

$$\begin{array}{r}x-3)7x^2-29x+24(7x-8\\ \underline{7x^2-21x}\\ -8x+24\\ \underline{-8x+24}\end{array}$$

Hence, $x-3$ is the H.C.F. required.

Note 6. If the given quantities have a common factor which can be seen by inspection, remove it, and find the H.C.F. of the resulting expressions. This result, multiplied by the common factor, will give the H.C.F. of the given quantities.

5. Find the H.C.F. of

$$6x^3-ax^2-5a^2x \text{ and } 21x^3-26ax^2+5a^2x.$$

Removing the common factor x, we find the H.C.F. of $6x^2-ax-5a^2$ and $21x^2-26ax+5a^2$. Multiplying the latter by 2,

$$\begin{array}{r}6x^2-ax-5a^2)42x^2-52ax+10a^2(7\\ \underline{42x^2-7ax-35a^2}\\ -45ax+45a^2\end{array}$$

Dividing by $-45a$,

$$\begin{array}{r}x-a)6x^2-ax-5a^2(6x+5a\\ \underline{6x^2-6ax}\\ 5ax-5a^2\\ \underline{5ax-5a^2}\end{array}$$

Multiplying $x-a$ by x, the common factor, we have $x(x-a)$ or x^2-ax as the H.C.F. of the given expressions.

EXAMPLES.

131. Find the highest common factors of the following:

1. x^2+x-6 and $2x^2-11x+14$.
2. $6x^2-7x-24$ and $12x^2+8x-15$.
3. $2a^2-5a+3$ and $4a^3-2a^2-9a+7$.
4. $24x^2+11ax-28a^2$ and $40x^2-51ax+14a^2$.
5. $8a^3-22a^2+5a$ and $6a^2b-23ab+20b$.
6. $x^3-5mx^2+4m^2x$ and $x^4-mx^3+3m^2x^2-3m^3x$.
7. $5m^2n^2+58mn^2+33n^2$ and $10m^3+31m^2-20m-21$.
8. $2a^4+3a^3x-9a^2x^2$ and $6a^3-17a^2x+14ax^2-3x^3$.
9. x^3-8 and $x^3-6x^2+11x-6$.
10. $2x^3-3x^2-x+1$ and $6x^3-x^2+3x-2$.
11. $8m^2-22mn+5n^2$ and $6m^4-29m^3n+43m^2n^2-20mn^3$.
12. $ax^3+2ax^2+ax+2a$ and $3x^5-12x^3-3x^2-6x$.
13. $ax^4-ax^3-2ax^2+2ax$ and $ax^5-3ax^4+2ax^3+ax^2-ax$.
14. $2x^4-2x^3+4x^2+2x+6$ and $3x^4+6x^3-3x-6$.
15. $a^4+a^3-6a^2+a+3$ and $a^4+2a^3-6a^2-a+2$.
16. $x^5-x^4-5x^3+2x^2+6x$ and $x^5+x^4-x^3-2x^2-2x$.
17. $15a^2x^3-20a^2x^2-65a^2x-30a^2$
    and $12bx^3+20bx^2-16bx-16b$.
18. $a^4+a^3x+a^2x^2+ax^3-4x^4$
    and $a^4+2a^3x+3a^2x^2+4ax^3-10x^4$.

19. $x^4+x^3+x^2-1$ and x^5+3x^4+2x.

20. $x^4-x^3y-3x^2y^2+5xy^3-6y^4$
and $3x^4-5x^3y-x^2y^2-7xy^3+10y^4$.

21. $2x^4-5x^3+5x^2-5x+3$ and $2x^4-7x^3+4x^2+5x-3$.

22. $3a^4-2a^3b+2a^2b^2-5ab^3-2b^4$
and $6a^4-a^3b+2a^2b^2-2ab^3-b^4$.

132. To find the H.C.F. of three or more quantities, find the H.C.F. of two of them; then of this result and the third quantity, and so on. The last divisor will be the H.C.F. of the given quantities.

EXAMPLES.

Find the highest common factors of the following:

1. $2x^2-5x-42$, $4x^2+8x-21$, and $6x^2+23x+7$.

2. $12x^2-28x-5$, $14x^2-39x+10$, and $10x^2-11x-35$.

3. $6m^2+7mn+2n^2$, $3m^3-7m^2n-12mn^2-4n^3$,
and $15m^2+4mn-4n^2$.

4. $6a^2+13a-5$, $6a^3+19a^2+8a-5$,
and $3a^3+2a^2+2a-1$.

5. x^3+3x^2-6x-8, x^3+5x^2+2x-8,
and $x^3-3x^2-16x+48$.

6. x^3-7x+6, $x^3+3x^2-16x+12$,
and x^3-5x^2+7x-3.

7. $2a^3-3a^2-5a+6$, $2a^3+3a^2-8a-12$,
and $2a^3-a^2-12a-9$.

X. LOWEST COMMON MULTIPLE.

133. A **Common Multiple** of two or more quantities is a quantity which can be divided by each of them without a remainder.

Hence, a common multiple of two or more quantities must contain all the prime factors of each of the quantities.

134. The **Lowest Common Multiple** of two or more quantities is the product of their different prime factors, each being taken the greatest number of times which it occurs in any one of the quantities.

It is evident from this definition that the lowest common multiple of two or more quantities is the expression of *lowest degree* which can be divided by each of them without a remainder.

Thus, the lowest common multiple of x^3y^2, y^5z, and x^2z^4 is $x^3y^5z^4$.

When quantities are prime to each other, their product is their lowest common multiple.

135. In determining the lowest common multiple of algebraic quantities, we may distinguish three cases.

Case I.

136. *When the quantities are monomials.*

1. Find the L.C.M. of $36a^3x$, $60a^2y^2$, and $84cx^3$.

$$36a^3x = 2\cdot2\cdot3\cdot3\cdot a^3x$$
$$60a^2y^2 = 2\cdot2\cdot3\cdot5\cdot a^2y^2$$
$$84cx^3 = 2\cdot2\cdot3\cdot7\cdot cx^3$$

Hence, the L.C.M. $= 2\cdot2\cdot3\cdot3\cdot5\cdot7\cdot a^3cx^3y^2$ (Art. 134)

$= 1260a^3cx^3y^2$, *Ans.*

RULE.

To the lowest common multiple of the coefficients, annex all the letters which occur in the given quantities, giving to each the highest exponent which it has in any of the quantities.

EXAMPLES.

Find the lowest common multiples of the following:

2. $6a^3b$, a^2b^2.

3. $10x^2y$, $12y^3z$.

4. $30m^2$, $27n^2$.

5. $6ab$, $10bc$, $14ca$.

6. a^5b^2, $9a^3b^4$, $12a^2b^3$.

7. $16x^2y$, $42y^3z$.

8. $8c^2d^3$, $10ac$, $18a^2d$.

9. $24m^3x^2$, $30n^2y$, $32xy^2$.

10. $36xy^2z^3$, $63x^3yz^2$, $28x^2y^3z$.

11. $40a^2bd^3$, $90ac^3d^4$, $54b^3cd^2$.

CASE II.

137. *When the quantities are polynomials which can be readily factored by inspection.*

1. Find the L.C.M. of x^2+x-6, x^2-4x+4, and x^3-9x.

$$x^2+x-6 = (x+3)(x-2)$$
$$x^2-4x+4 = (x-2)^2$$
$$x^3-9x = x(x+3)(x-3)$$

Hence, the L.C.M. $= x(x-2)^2(x+3)(x-3)$, (Art. 134)
$= x(x-2)^2(x^2-9)$, *Ans.*

EXAMPLES.

Find the lowest common multiples of the following:

2. x^2-y^2 and $xy-y^2$.

3. x^2-1 and x^2-7x-8.

4. $8a^2b + 8ab^2$ and $6a - 6b$.

5. $m^2 - n^2$ and $m^3 - n^3$.

6. $a - b$ and $a^2 - 4ab + 3b^2$.

7. $x^2 - 2xy + y^2$ and $x^3y - xy^3$.

8. $2a^2 + 2ab$, $3ab - 3b^2$, and $4a^2c - 4b^2c$.

9. $x^2 + 2ax - 35a^2$ and $x^2 - 2ax - 15a^2$.

10. $mn + n^2$, $mn - n^2$, and $m^2 - n^2$.

11. $ax - 2a + bx - 2b$ and $a^2 - 2ab - 3b^2$.

12. $ax^2 + a^2x$, $x^2 - a^2$, and $x^3 - a^3$.

13. $8(a^2 - b^2)$, $6(a + b)^2$, and $12(a - b)^2$.

14. $x^3 - 10x^2 + 21x$ and $ax^2 + 5ax - 24a$.

15. $x^2 - 1$, $x^2 - 2x + 1$, and $x^2 + 2x + 1$.

16. $2 - 2x^2$, $4 - 4x$, $8 + 8x$, and $12 + 12x^2$.

17. $x^2 + 5x + 4$, $x^2 + 2x - 8$, and $x^2 + 7x + 12$.

18. $a(x - b)(x - c)$, $b(x - c)(x - a)$, and $c(x - a)(x - b)$.

19. $(2m - 1)^2$, $4m^2 - 1$, and $8m^3 - 1$.

20. $a^2 + a$, $a^4 - a^2$, and $a^6 + a^3$.

21. $a^2 - 4a + 3$, $a^2 + a - 12$, and $a^2 - a - 20$.

22. $1 - x^4$, $1 + 2x^2 + x^4$, and $1 - 2x^2 + x^4$.

23. $(a + b)^2 - c^2$ and $(a - c)^2 - b^2$.

24. $ax - ay - bx + by$, $(x - y)^2$, and $3a^2b - 3ab^2$.

25. $9x^3 + 12x^2 + 4x$, $18ax^4 - 12ax^3 + 8ax^2$, and $27x^3 + 8$.

26. $x^2 - y^2 - z^2 + 2yz$ and $x^2 - y^2 + z^2 + 2xz$.

Case III.

138. *When the quantities are polynomials which cannot be readily factored by inspection.*

Let A and B be two expressions; let F be their highest common factor, and M their lowest common multiple. Suppose that $A = aF$ and $B = bF$; then,

$$A \times B = abF^2 \qquad (1)$$

Since a and b can have no common factor, the L.C.M. of aF and bF is abF; that is, $M = abF$; whence,

$$F \times M = abF^2 \qquad (2)$$

From (1) and (2) we have $A \times B = F \times M$ (Art. 42, 7).

That is, *the product of any two quantities is equal to the product of their highest common factor and lowest common multiple.*

Hence, to find the L.C.M. of two quantities,

Divide their product by their highest common factor; or,

Divide one of the quantities by their highest common factor, and multiply the quotient by the other quantity.

139. 1. Find the L.C.M. of

$$6x^2 - 17x + 12 \text{ and } 12x^2 - 4x - 21.$$

$$\begin{array}{l}
6x^2 - 17x + 12)\,12x^2 - 4x - 21\,(2 \\
\qquad\qquad\qquad\quad \underline{12x^2 - 34x + 24} \\
\qquad\qquad\qquad\qquad\quad 30x - 45 \\
\qquad\qquad\qquad\qquad\quad 2x - 3)\,6x^2 - 17x + 12\,(3x - 4 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{6x^2 - 9x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad - 8x + 12 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{- 8x + 12}
\end{array}$$

That is, the H.C.F. of the quantities is $2x - 3$. Dividing $6x^2 - 17x + 12$ by $2x - 3$, the quotient is $3x - 4$.

Hence, the L.C.M. $= (3x - 4)(12x^2 - 4x - 21)$

$$= 36x^3 - 60x^2 - 47x + 84, \textit{ Ans.}$$

EXAMPLES.

Find the lowest common multiples of the following:

2. $2x^2+x-6$ and $4x^2-8x+3$.

3. $6x^2+13x-28$ and $12x^2-31x+20$.

4. $8x^2+30x+7$ and $12x^2-29x-8$.

5. $6x^3-8x^2-30x$ and $6ax^2+19ax+15a$.

6. $a^2-8ab+7b^2$ and $a^3-9a^2b+23ab^2-15b^3$.

7. $2m^2n-3mn-2n$ and $2m^4-6m^3+6m^2-8m+8$.

8. $6ax^2-a^2x-12a^3$ and $10ax^2-17a^2x+3a^3$.

9. a^3+a^2-8a-6 and $2a^3-5a^2-2a+2$.

10. $2x^3+x^2-x+3$ and $2x^3+5x^2-x-6$.

11. $a^3-2a^2b+2ab^2-b^3$ and $a^3+a^2b-ab^2-b^3$.

12. $x^4+2x^3+2x^2+x$ and $ax^3-2ax-a$.

13. $2x^4-11x^3+3x^2+10x$ and $3x^4-14x^3-6x^2+5x$.

14. x^4-x^3-8x+8 and x^4-8x^2+9x-2.

140. To find the L.C.M. of three or more quantities, find the L.C.M. of two of them; then of this result and the third quantity; and so on.

EXAMPLES.

Find the lowest common multiples of the following:

1. x^2-1, $2x^2-9x+7$, and $2x^2+3x-5$.

2. $3a^2-2a-1$, $6a^2-a-1$, and $9a^2-3a-2$.

3. $2x^2-5x+2$, $4x^2+4x-3$, and $10x^2-7x+1$.

4. $4x^2-6x-18$, $4x^3+4x^2-3x$, and $6x^4+5x^3-6x^2$.

5. $a^3-6a^2+11a-6$, $a^3-a^2-14a+24$,
and $a^3+a^2-17a+15$.

XI. FRACTIONS.

141. The expression $\frac{a}{b}$ signifies $a \div b$; in other words, $\frac{a}{b}$ denotes that a units are divided into b equal parts, and that *one* part is taken.

Or, what is the same thing, $\frac{a}{b}$ denotes that *one* unit is divided into b equal parts, and that a parts are taken.

142. The expression $\frac{a}{b}$ is called a **Fraction**; a is called the *numerator*, and b the *denominator*.

By Art. 141, the denominator shows into how many parts the unit is divided, and the numerator shows how many parts are taken.

The numerator and denominator are called the *terms* of the fraction.

143. An **Entire Quantity** or **Integer** is one which has no fractional part; as $2xy$, or $a + b$.

Every integer may be considered as a fraction whose denominator is unity; thus, $a = \frac{a}{1}$.

144. A **Mixed Quantity** is one having both entire and fractional parts; as $a + \frac{b}{2}$, or $x + \frac{a}{y+z}$.

GENERAL PRINCIPLES.

145. *If the numerator of a fraction be multiplied, or the denominator divided, by any quantity, the fraction is multiplied by that quantity.*

I. Let $\frac{a}{b}$ be any fraction. Multiplying its numerator by c, we have $\frac{ac}{b}$. To prove that $\frac{ac}{b}$ is c times $\frac{a}{b}$.

In each of these fractions the unit is divided into b equal parts; in the first case ac parts are taken, and in the second case a parts. Since c times as many parts are taken in $\frac{ac}{b}$ as in $\frac{a}{b}$, it follows that

$$\frac{ac}{b} = c \times \frac{a}{b}. \tag{1}$$

II. Let $\frac{a}{bc}$ be any fraction. Dividing its denominator by c, we have $\frac{a}{b}$. To prove that $\frac{a}{b}$ is c times $\frac{a}{bc}$.

In each of these fractions a parts are taken; but since in the first case the unit is divided into b equal parts, and in the second case into bc equal parts, the parts in $\frac{a}{b}$ will be c times as great as in $\frac{a}{bc}$. Hence,

$$\frac{a}{b} = c \times \frac{a}{bc}. \tag{2}$$

146. *If the numerator of a fraction be divided, or the denominator multiplied, by any quantity, the fraction is divided by that quantity.*

I. Let $\frac{ac}{b}$ be any fraction. Dividing its numerator by c, we have $\frac{a}{b}$. To prove that $\frac{a}{b}$ is $\frac{ac}{b}$ divided by c.

By Art. 145, (1), $\qquad c \times \frac{a}{b} = \frac{ac}{b}$.

Whence it follows that $\qquad \frac{a}{b} = \frac{ac}{b} \div c$.

II. Let $\frac{a}{b}$ be any fraction. Multiplying its denominator by c, we have $\frac{a}{bc}$. To prove that $\frac{a}{bc}$ is $\frac{a}{b}$ divided by c.

By Art. 145, (2), $$c \times \frac{a}{bc} = \frac{a}{b}.$$

Whence it follows that $$\frac{a}{bc} = \frac{a}{b} \div c.$$

147. *If the numerator and denominator of a fraction be both multiplied, or both divided, by the same quantity, the value of the fraction is not altered.*

For, by Arts. 145 and 146, multiplying the numerator multiplies the fraction, and multiplying the denominator divides it. Hence, the fraction is both multiplied and divided by the same quantity, and its value is not altered.

Similarly we may show that if both terms are divided by the same quantity, the value of the fraction is not altered.

TO REDUCE A FRACTION TO ITS LOWEST TERMS.

148. A fraction is in its *lowest terms* when its numerator and denominator are prime to each other.

Case I.

149. *When the numerator and denominator can be readily factored by inspection.*

Since dividing both numerator and denominator by the same quantity, or canceling equal factors in each, does not alter the value of the fraction (Art. 147), we have the following rule:

Resolve both numerator and denominator into their prime factors, and cancel all which are common to both.

1. Reduce $\frac{18a^3b^2c}{45a^2b^2x}$ to its lowest terms.

$$\frac{18a^3b^2c}{45a^2b^2x}=\frac{2\cdot3\cdot3\cdot a^3b^2c}{3\cdot3\cdot5\cdot a^2b^2x}.$$

Dividing both terms by $3\cdot3\cdot a^2b^2$, we have $\frac{2ac}{5x}$, *Ans.*

2. Reduce $\frac{x^3-27}{x^2-2x-3}$ to its lowest terms.

$$\frac{x^3-27}{x^2-2x-3}=\frac{(x-3)(x^2+3x+9)}{(x-3)(x+1)}=\frac{x^2+3x+9}{x+1},\ \textit{Ans.}$$

Note. If all the factors of the numerator be removed by cancellation, unity (which is a factor of all algebraic expressions) remains to form a numerator.

If all the factors of the denominator be removed, the result is an entire quantity; this being a case of exact division.

EXAMPLES.

3. $\frac{x^4y^2z}{xy^2z^3}$.

4. $\frac{2a^2b^5c}{5a^3bc^3}$.

5. $\frac{12xy^2}{32x^3}$.

6. $\frac{32mn}{56m^4n^3}$.

7. $\frac{65x^2y^5z^4}{26x^4y^3z^2}$.

8. $\frac{54a^3c^2}{72a^2bc}$.

9. $\frac{15mxy^2}{75mx^2y^3}$.

10. $\frac{115c^3x^2y}{23c^2x^2}$.

11. $\frac{154m^2x^3}{88m^3xy^2}$.

12. $\frac{2a^2cd+2abcd}{6a^2xy+6abxy}$.

13. $\frac{3x^5-6x^4y}{6x^2y^2-12xy^3}$.

14. $\frac{2x^3y-6x^2y}{x^2-8x+15}$.

15. $\frac{a^2-2a-15}{a^2+10a+21}$.

16. $\frac{6a^3b+3a^2b^2}{3a^2b^2+6ab^3}$.

17. $\frac{4c^2-20c+25}{4c^3-25c}$.

18. $\frac{m^2-10m+16}{m^2+m-72}$.

19. $\frac{9an^2-4a}{9bn^2-12bn+4b}$.

20. $\frac{a^2-4b^2}{a^2+ab-6b^2}$.

21. $\frac{8x^3+y^3}{4x^3-2x^2y+xy^2}$.

22. $\frac{ac-ad-bc+bd}{a^3-b^3}$.

23. $\frac{ax^2-4a}{x^3-9x^2+14x}$.

24. $\frac{27y^3-125}{9y^2-30y+25}$.

25. $\frac{x^3-x^2+2x-2}{2x^3+x^2+4x+2}$.

26. $\frac{x^2-4x+16}{ax^4+64ax}$.

27. $\frac{a^2-(b+c)^2}{(a-b)^2-c^2}$.

28. $\frac{(x^2-4)(x^2-3x+2)}{(x^2-4x+4)(x^2+x-2)}$.

29. $\frac{(a-b)^2-(c-d)^2}{(a-c)^2-(b-d)^2}$.

Case II.

150. *When the numerator and denominator cannot be readily factored by inspection.*

Since the H.C.F. of two quantities is the product of their common prime factors, we have the following rule:

Divide both numerator and denominator by their highest common factor.

EXAMPLES.

1. Reduce $\frac{2a^2-5a+3}{6a^2-a-12}$ to its lowest terms.

By the rule of Art. 130, the H.C.F. of $2a^2-5a+3$ and $6a^2-a-12$ is $2a-3$. Dividing the numerator by $2a-3$, the quotient is $a-1$; and dividing the denominator, the quotient is $3a+4$. Hence,

$$\frac{2a^2-5a+3}{6a^2-a-12}=\frac{a-1}{3a+4}, \quad \textit{Ans.}$$

Reduce the following to their lowest terms:

2. $\dfrac{x^2-6x+5}{3x^2+4x-7}$.

3. $\dfrac{10a^2-a-21}{2a^2-7a+6}$.

4. $\dfrac{2m^2-5m+3}{12m^2-28m+15}$.

5. $\dfrac{x^2-2x-3}{x^3-2x^2-2x-3}$.

6. $\dfrac{12m^2+16mn-3n^2}{10m^2+mn-21n^2}$.

7. $\dfrac{x^3+x^2-3x-2}{x^3-4x^2+2x+3}$.

8. $\dfrac{6x^3-7x^2+5x-2}{2x^3+5x^2-2x+3}$.

9. $\dfrac{6y^3-19y^2+7y+12}{6y^3-25y^2+17y+20}$.

10. $\dfrac{a^3-3a^2+a+2}{2a^3-3a^2-a-2}$.

11. $\dfrac{x^3-4x^2y+4xy^2-y^3}{x^3-2x^2y+4xy^2-3y^3}$.

151. Since a fraction represents the quotient of its numerator divided by its denominator, it is positive when its terms have the same sign, and negative when they have different signs.

Thus, if $\dfrac{a}{b}=x$,

then $$\frac{-a}{-b}=x, \text{ and } \frac{-a}{b}=\frac{a}{-b}=-x.$$

152. It follows from Art. 151 that the fraction $\dfrac{a}{b}$ can be written in any one of the forms

$$\frac{-a}{-b},\ -\frac{-a}{b},\text{ or }-\frac{a}{-b}.$$

That is, *if the signs of both numerator and denominator are changed, the value of the fraction is not altered. But if the sign of either one is changed, the sign before the fraction is changed.*

153. If either numerator or denominator is a polynomial, care must be taken, on changing its sign, to change the sign of *each of its terms*.

Thus, the fraction $\frac{a-b}{c-d}$, by changing the signs of both numerator and denominator, can be written in the form $\frac{-(a-b)}{-(c-d)}$, or $\frac{b-a}{d-c}$ (Art. 67).

154. It follows from Art. 151 that the fraction $\frac{ab}{cd}$ can be written in any one of the forms

$$\frac{(-a)b}{c(-d)}, \quad \frac{(-a)(-b)}{cd}, \quad \frac{(-a)(-b)}{(-c)(-d)}, \text{ etc.};$$

or,

$$-\frac{(-a)b}{cd}, \quad -\frac{ab}{(-c)d}, \quad -\frac{(-a)(-b)}{c(-d)}, \text{ etc.}$$

From which it appears that

If the terms of a fraction are composed of factors, the signs of any even number of factors may be changed without altering the value of the fraction. But if the signs of any odd number of factors are changed, the sign before the fraction is changed.

Thus, the fraction $\frac{a-b}{(x-y)(x-z)}$ can be written in any one of the forms

$$\frac{a-b}{(y-x)(z-x)}, \quad \frac{b-a}{(y-x)(x-z)}, \quad -\frac{b-a}{(y-x)(z-x)}, \text{ etc.}$$

TO REDUCE A FRACTION TO AN ENTIRE OR MIXED QUANTITY.

155. Since a fraction is an expression of division, we have the following rule:

Divide the numerator by the denominator.

1. Reduce $\dfrac{6x^2-15x-2}{3x}$ to a mixed quantity.

Dividing each term of the numerator by the denominator,

$$\frac{6x^2-15x-2}{3x}=\frac{6x^2}{3x}-\frac{15x}{3x}-\frac{2}{3x}=2x-5-\frac{2}{3x}, \textit{ Ans.}$$

2. Reduce $\dfrac{8x^3-12x^2-9x+10}{4x^2-3}$ to a mixed quantity.

$$\begin{array}{r|l}
4x^2-3)\,8x^3-12x^2-9x+10 & (2x-3 \\
\underline{8x^3\qquad\quad-6x\qquad} & \\
-12x^2-3x+10 & \\
\underline{-12x^2\qquad\quad+9} & \\
-3x+1 &
\end{array}$$

A remainder whose first term will not contain the first term of the divisor, may be written over the divisor in the form of a fraction, and added to the quotient. Thus, the result is

$$2x-3+\frac{-3x+1}{4x^2-3}.$$

Or, since the sign of each term of the numerator may be changed, if at the same time the sign before the fraction is changed (Art. 152), we have

$$\frac{8x^3-12x^2-9x+10}{4x^2-3}=2x-3-\frac{3x-1}{4x^2-3}, \textit{ Ans.}$$

EXAMPLES.

Reduce the following to mixed quantities:

3. $\dfrac{5x^2-10x+4}{5x}$.

4. $\dfrac{6x^3-3x^2+9x-7}{3x}$.

5. $\dfrac{x^3+2y^3}{x+y}$.

6. $\dfrac{2x^2-41}{x-3}$.

7. $\dfrac{a^3-a^2-a-2}{a^2+a-1}$.

8. $\dfrac{12x^2-8x+7}{4x-1}$.

9. $\dfrac{a^4+b^4}{a+b}$.

10. $\dfrac{4m^3-16m^2n+29mn^2-22n^3}{2m-3n}$.

11. $\dfrac{2a^4-a^3-9a^2+14}{2a^2-a-3}$.

12. $\dfrac{x^3+2x^2+3x+4}{x^2+x+1}$.

13. $\dfrac{x^5-y^5}{x+y}$.

14. $\dfrac{6x^3-13x^2+6x-6}{3x^2-2x+1}$.

TO REDUCE A MIXED QUANTITY TO A FRACTIONAL FORM.

156. The operation being the converse of that of Art. 155, we have the following rule:

Multiply the integral part by the denominator; add the numerator to the product when the sign before the fraction is +, *and subtract it when the sign is* −; *and write the result over the denominator.*

1. Reduce $\dfrac{x-5}{2x-3}+x-2$ to a fractional form.

By the rule,

$$\frac{x-5}{2x-3}+x-2=\frac{x-5+(x-2)(2x-3)}{2x-3}$$

$$=\frac{x-5+2x^2-7x+6}{2x-3}$$

$$=\frac{2x^2-6x+1}{2x-3},\ \textit{Ans.}$$

2. Reduce $a+b-\dfrac{a^2-b^2-5}{a-b}$ to a fractional form.

$$a+b-\frac{a^2-b^2-5}{a-b}=\frac{(a+b)(a-b)-(a^2-b^2-5)}{a-b}$$

$$=\frac{a^2-b^2-a^2+b^2+5}{a-b}=\frac{5}{a-b},\ \textit{Ans.}$$

Note. If the numerator is a polynomial, it will be found convenient to enclose it in a parenthesis, when the sign before the fraction is −.

EXAMPLES.

Reduce the following to fractional forms:

3. $x+1+\dfrac{x+1}{x}$.

4. $x+1-\dfrac{4}{x+3}$.

5. $\dfrac{2m^2-3n^2}{3m+n}+m-n$.

6. $7x-3-\dfrac{53x-20}{8}$.

7. $1-\dfrac{m-n}{m+n}$.

8. $a+b-\dfrac{a^2+b^2}{a+b}$.

9. $\dfrac{2}{2x+1}+3x-2$.

10. $a^2-b^2+\dfrac{ab(a+b)}{a-b}$.

11. $\dfrac{x+y}{x-y}-1$.

12. $m-n+\dfrac{m^2+n^2}{m+n}$.

13. $a^2-ab+b^2-\dfrac{2b^3}{a+b}$.

14. $x^2-3x-\dfrac{2x(3-x)}{x-2}$.

15. $\dfrac{m^3+n^3}{m^2+mn+n^2}-(m-n)$.

16. $1+2x+4x^2+\dfrac{x^3+1}{2x-1}$.

17. $x-2y-\dfrac{x^3-8y^3}{x^2-4xy+4y^2}$.

18. $x^2-2x+3-\dfrac{x^3+13x-5}{x^2+3x-2}$.

TO REDUCE FRACTIONS TO THEIR LOWEST COMMON DENOMINATOR.

157. **1.** Reduce $\dfrac{5cd}{3a^2b}$, $\dfrac{3mx}{2ab^2}$, and $\dfrac{3ny}{4a^3b}$ to equivalent fractions having the lowest common denominator.

The lowest common denominator is the lowest common multiple of $3a^2b$, $2ab^2$, and $4a^3b$, which is $12a^3b^2$.

By Art. 147, both terms of a fraction may be multiplied by the same quantity without altering its value. Hence,

Multiplying both terms of $\frac{5\,cd}{3\,a^2b}$ by $4\,ab$, we have $\frac{20\,abcd}{12\,a^3b^2}$.

Multiplying both terms of $\frac{3\,mx}{2\,ab^2}$ by $6\,a^2$, we have $\frac{18\,a^2mx}{12\,a^3b^2}$.

Multiplying both terms of $\frac{3\,ny}{4\,a^3b}$ by $3\,b$, we have $\frac{9\,bny}{12\,a^3b^2}$.

Therefore the required fractions are

$$\frac{20\,abcd}{12\,a^3b^2},\quad \frac{18\,a^2mx}{12\,a^3b^2},\quad \text{and}\quad \frac{9\,bny}{12\,a^3b^2},\quad \textit{Ans.}$$

It will be observed that the terms of each fraction are multiplied by a quantity which is obtained by dividing the lowest common denominator by its own denominator. Hence the following rule:

Find the lowest common multiple of the given denominators. Divide this by each denominator separately, multiply the corresponding numerators by the quotients, and write the results over the common denominator.

Note. Before applying the rule, each fraction should be in its lowest terms.

EXAMPLES.

Reduce the following to equivalent fractions having the lowest common denominator:

2. $\frac{3\,ab}{14}$, $\frac{2\,ac}{21}$, and $\frac{5\,bc}{6}$.

3. $\frac{2}{a^3x^2}$, $\frac{3}{ax^3}$, and $\frac{4}{a^2x}$.

4. $\frac{4\,c-1}{8\,ab^2}$ and $\frac{3\,b-2}{12\,a^2c}$.

5. $\frac{5\,az}{6\,x^2y}$, $\frac{3\,bx}{8\,y^2z}$, and $\frac{7\,cy-m}{10\,xz^2}$.

6. $\dfrac{2a}{a^2+a-6}$ and $\dfrac{4a}{a^2-4}$.

7. $\dfrac{1}{x^2-1}$ and $\dfrac{1}{x^3-1}$.

8. m, $\dfrac{m^3}{mn-n^2}$, and $\dfrac{mn^2}{m^2-n^2}$.

9. $\dfrac{2}{a-b}$, $\dfrac{3}{a+b}$, and $\dfrac{4}{a^2+b^2}$.

10. $\dfrac{ay}{1-x}$, $\dfrac{ax^2}{(1-x)^2}$, and $\dfrac{xy^3}{(1-x)^3}$.

11. $\dfrac{ab}{am-bm+an-bn}$ and $\dfrac{m-n}{2a^2-2ab}$.

12. $\dfrac{x+3}{x^2-3x+2}$, $\dfrac{x+1}{x^2-5x+6}$, and $\dfrac{x+2}{x^2-4x+3}$.

ADDITION AND SUBTRACTION OF FRACTIONS.

158. It follows from the definition of Art. 141 that

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}, \text{ and } \frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}.$$

Hence the following

RULE.

To add fractions, reduce them, if necessary, to equivalent fractions having the lowest common denominator. Add the numerators of the resulting fractions, and write the sum over the common denominator.

To subtract one fraction from another, reduce them to equivalent fractions having the lowest common denominator. Subtract the numerator of the subtrahend from that of the minuend, and write the result over the common denominator.

Note. The final result should be reduced to its simplest form.

1. Required the sum of $\frac{4a-1}{4ac}$ and $\frac{3-5b^2}{6b^2c}$.

The lowest common denominator is $12ab^2c$. Multiplying the terms of the first fraction by $3b^2$, and of the second by $2a$, we have

$$\frac{4a-1}{4ac}+\frac{3-5b^2}{6b^2c}=\frac{12ab^2-3b^2}{12ab^2c}+\frac{6a-10ab^2}{12ab^2c}$$

$$=\frac{12ab^2-3b^2+6a-10ab^2}{12ab^2c}$$

$$=\frac{2ab^2-3b^2+6a}{12ab^2c},\ \textit{Ans.}$$

2. Subtract $\frac{4x-1}{2x}$ from $\frac{6a-2}{3a}$.

The L.C.D. is $6ax$. Hence,

$$\frac{6a-2}{3a}-\frac{4x-1}{2x}=\frac{12ax-4x}{6ax}-\frac{12ax-3a}{6ax}$$

$$=\frac{12ax-4x-(12ax-3a)}{6ax}$$

$$=\frac{12ax-4x-12ax+3a}{6ax}$$

$$=\frac{3a-4x}{6ax},\ \textit{Ans.}$$

Note. If a fraction, whose numerator is a polynomial, is preceded by a $-$ sign, care must be taken to change the sign of each term in the numerator before combining it with the others. It is convenient in such a case to enclose the numerator in a parenthesis, as shown in Ex. 2.

EXAMPLES.

Simplify the following:

3. $\frac{2x-5}{12}+\frac{3x+11}{18}$.

4. $\frac{3}{5ab^2}-\frac{1}{2a^2b}$.

5. $\frac{2a+3}{6}-\frac{3a+5}{8}$.

7. $\frac{b-4a}{24a}+\frac{a+5b}{30b}$.

6. $\frac{m-2}{2mn}-\frac{2-3mn^2}{3m^2n^3}$.

8. $\frac{a-b}{4}+\frac{2a+b}{6}-\frac{3a-b}{8}$.

9. $\frac{a^2+1}{3a^2}-\frac{6a^3+1}{12a^3}+\frac{b-2}{6b}$.

10. $\frac{2x-1}{12}+\frac{2x+3}{15}-\frac{6x+1}{20}$.

11. $\frac{m+2}{7}-\frac{m+2}{14}-\frac{m+3}{21}$.

12. $\frac{2}{3}-\frac{2x-1}{6x}-\frac{3x^2+1}{9x^2}$.

13. $\frac{x-2}{2}+\frac{3x+1}{3}-\frac{6x-5}{4}-\frac{3}{5}$.

14. $\frac{3a+1}{12a}-\frac{2b-1}{8b}+\frac{4c-1}{16c}-\frac{6d+1}{24d}$.

15. Simplify $\frac{1}{x+x^2}+\frac{1}{x-x^2}$.

The L.C.M. of $x+x^2$ and $x-x^2$ is $x(1+x)(1-x)$, or $x(1-x^2)$. Multiplying the terms of the first fraction by $1-x$, and of the second by $1+x$, we have

$$\frac{1}{x+x^2}+\frac{1}{x-x^2}=\frac{1-x}{x(1-x^2)}+\frac{1+x}{x(1-x^2)}$$

$$=\frac{1-x+1+x}{x(1-x^2)}$$

$$=\frac{2}{x(1-x^2)}, \textit{ Ans.}$$

16. Simplify $\frac{a+b}{a-b}-\frac{a-b}{a+b}-\frac{4ab}{a^2-b^2}$.

The L.C.D. is a^2-b^2. Hence,

$$\frac{a+b}{a-b}-\frac{a-b}{a+b}-\frac{4ab}{a^2-b^2}$$

$$=\frac{(a+b)^2}{a^2-b^2}-\frac{(a-b)^2}{a^2-b^2}-\frac{4ab}{a^2-b^2}$$

$$=\frac{(a+b)^2-(a-b)^2-4ab}{a^2-b^2}$$

$$=\frac{a^2+2ab+b^2-(a^2-2ab+b^2)-4ab}{a^2-b^2}$$

$$=\frac{a^2+2ab+b^2-a^2+2ab-b^2-4ab}{a^2-b^2}=0,\ \textit{Ans.}$$

Simplify the following:

17. $\frac{1}{1+x}+\frac{1}{1-x}$.

18. $\frac{1}{x+2}+\frac{1}{3-x}$.

19. $\frac{1}{x+7}-\frac{1}{x+8}$.

20. $\frac{a}{a-b}-\frac{b}{a+b}$.

21. $\frac{a+b}{a-b}-\frac{a-b}{a+b}$.

22. $\frac{x+y}{y}-\frac{2xy+x^2}{y(x+y)}$.

23. $\frac{1+x}{1-x}-\frac{1-x}{1+x}$.

24. $\frac{m+n}{(m-n)^2}+\frac{2m}{m^2-n^2}$.

25. $\frac{1}{a^2-4a+4}-\frac{1}{a^2+a-6}$.

26. $\frac{x}{x-y}+\frac{3x}{x+y}-\frac{2xy}{x^2-y^2}$.

27. $\frac{a}{a+b}+\frac{b}{a-b}+\frac{2ab}{a^2-b^2}$.

28. $\frac{m}{mn-n^2}-\frac{1}{m-n}-\frac{1}{n}$.

29. $\frac{x+y}{x-y}+\frac{x-y}{x+y}+2$.

30. $\frac{2}{x}-\frac{3}{2x-1}-\frac{2x-3}{4x^2-1}$.

31. $\dfrac{1}{a+b}+\dfrac{1}{a-b}-\dfrac{2a}{a^2+b^2}$. **32.** $\dfrac{1}{1-x}-\dfrac{x}{(1-x)^2}-\dfrac{x^2-4x}{(1-x)^3}$.

33. $\dfrac{1}{ab-cd}-\dfrac{1}{ab+cd}-\dfrac{2cd}{a^2b^2-c^2d^2}$.

34. $\dfrac{x-3}{x-2}-\dfrac{x+1}{x+5}+\dfrac{x+13}{x^2+3x-10}$.

35. $\dfrac{x-a}{x-b}+\dfrac{x-b}{x-a}-\dfrac{(a-b)^2}{(x-a)(x-b)}$.

36. $\dfrac{1}{x(x+1)}-\dfrac{1}{x(x-1)}+\dfrac{x}{x^2-1}$.

37. $\dfrac{a+b}{(b-c)(c-a)}+\dfrac{b+c}{(c-a)(a-b)}+\dfrac{c+a}{(a-b)(b-c)}$.

38. $\dfrac{1}{x-1}-\dfrac{x}{x^2-1}+\dfrac{3}{x^3-1}$.

39. $\dfrac{2x-6}{x^2+3x+2}-\dfrac{x+2}{x^2-2x-3}-\dfrac{x+1}{x^2-x-6}$.

40. $\dfrac{x-y}{(x+z)(y+z)}+\dfrac{y-z}{(x+y)(x+z)}-\dfrac{z-x}{(x+y)(y+z)}$.

In certain cases, the principles of Arts. 152 and 154 enable us to change the form of a fraction to one which is more convenient for the purposes of addition and subtraction.

41. Simplify $\dfrac{3}{a-b}+\dfrac{2b+a}{b^2-a^2}$.

Changing the signs of the terms in the denominator of the second fraction, and at the same time changing the sign before the fraction (Art. 152), we have

$$\frac{3}{a-b}-\frac{2b+a}{a^2-b^2}.$$

The L.C.D. is now a^2-b^2. Hence

$$\frac{3}{a-b}-\frac{2b+a}{a^2-b^2}=\frac{3(a+b)}{a^2-b^2}-\frac{2b+a}{a^2-b^2}$$

$$=\frac{3(a+b)-(2b+a)}{a^2-b^2}$$

$$=\frac{3a+3b-2b-a}{a^2-b^2}=\frac{2a+b}{a^2-b^2},\quad \textit{Ans.}$$

42. Simplify

$$\frac{1}{(x-y)(x-z)}-\frac{1}{(y-x)(y-z)}-\frac{1}{(z-x)(z-y)}.$$

By Art. 154, we may change the sign of the factor $y-x$ in the second denominator, at the same time changing the sign before the fraction; and we may change the signs of both factors of the third denominator. The expression then becomes

$$\frac{1}{(x-y)(x-z)}+\frac{1}{(x-y)(y-z)}-\frac{1}{(x-z)(y-z)}.$$

The L.C.D. is now $(x-y)(x-z)(y-z)$. Hence the result

$$=\frac{(y-z)+(x-z)-(x-y)}{(x-y)(x-z)(y-z)}=\frac{y-z+x-z-x+y}{(x-y)(x-z)(y-z)}$$

$$=\frac{2y-2z}{(x-y)(x-z)(y-z)}=\frac{2(y-z)}{(x-y)(x-z)(y-z)}$$

$$=\frac{2}{(x-y)(x-z)},\quad \textit{Ans.}$$

Simplify the following:

43. $\dfrac{4}{a^2-ab}+\dfrac{3}{b^2-ab}.$

44. $\dfrac{5a+1}{3a-3}-\dfrac{3a-1}{2-2a}.$

45. $\dfrac{1}{3x-x^2}+\dfrac{1}{x^2-9}.$

46. $\dfrac{1}{m^2-mn}-\dfrac{1}{n^2-m^2}.$

47. $\dfrac{1}{(a-2)(x+2)}+\dfrac{1}{(2-a)(x+a)}.$

48. $\dfrac{a}{a+b}+\dfrac{a}{b-a}+\dfrac{2a^2}{a^2-b^2}.$

49. $\dfrac{x}{1+x}-\dfrac{x}{1-x}-\dfrac{x^2}{x^2-1}.$

50. $\dfrac{3}{2-x}+\dfrac{5}{x-3}+\dfrac{1}{x^2-5x+6}.$

51. $\dfrac{1}{(a-b)(b-c)}+\dfrac{1}{(b-a)(a-c)}-\dfrac{1}{(c-a)(c-b)}.$

52. $\dfrac{2}{(x-2)(x-3)}-\dfrac{3}{(3-x)(4-x)}-\dfrac{1}{(x-4)(2-x)}.$

MULTIPLICATION OF FRACTIONS.

159. Required the product of $\frac{a}{b}$ and $\frac{c}{d}$.

In Arithmetic, $\frac{2}{3}$ times $\frac{5}{7}$ signifies two-thirds of $\frac{5}{7}$.

Similarly, in Algebra, $\frac{a}{b}\times\frac{c}{d}$ signifies a $b$ths of $\frac{c}{d}$. That is, *we divide* $\frac{c}{d}$ *by* b, *and multiply the result by* a.

By Art. 146, II., $\qquad \dfrac{c}{d}\div b=\dfrac{c}{bd}.$

By Art. 145, I., $\qquad \dfrac{c}{bd}\times a=\dfrac{ac}{bd}.$

Hence, $\qquad \dfrac{a}{b}\times\dfrac{c}{d}=\dfrac{ac}{bd}.$

We have therefore the following rule for the multiplication of fractions:

Multiply the numerators together for the numerator of the product, and the denominators for its denominator.

Mixed quantities should be reduced to a fractional form before applying the rule.

Common factors in the numerators and denominators should be canceled before performing the multiplication.

EXAMPLES.

1. Multiply $\frac{10a^3y}{9bx^2}$ by $\frac{3b^4x^3}{4a^3y^2}$.

$$\frac{10a^3y}{9bx^2} \times \frac{3b^4x^3}{4a^3y^2} = \frac{10 \cdot 3 \cdot a^3b^4x^3y}{9 \cdot 4 \cdot a^3bx^2y^2}$$

$$= \frac{5b^3x}{6y}, \text{ Ans.}$$

2. Multiply together $\frac{x^2-2x}{x^2-2x-3}$, $\frac{x^2-9}{x^2-x}$, and $\frac{x^2+x}{x^2+x-6}$.

$$\frac{x^2-2x}{x^2-2x-3} \times \frac{x^2-9}{x^2-x} \times \frac{x^2+x}{x^2+x-6}$$

$$= \frac{x(x-2)}{(x-3)(x+1)} \times \frac{(x+3)(x-3)}{x(x-1)} \times \frac{x(x+1)}{(x+3)(x-2)}$$

$$= \frac{x}{x-1}, \text{ Ans.}$$

Multiply the following:

3. $\frac{5a^2bc}{12mn^2}$ and $3mn$.

4. $\frac{3abx^2}{5ay^2}$ and $\frac{5xy^2}{3abx^3}$.

5. $\frac{2a}{3b}$, $\frac{6c}{5a}$, and $\frac{5b}{8c}$.

6. $\frac{8x^2}{9y^3}$, $\frac{15y^2}{16z^3}$, and $\frac{3z^4}{10x^3}$.

7. $\dfrac{3ab^2}{4cd}$, $\dfrac{3ac^2}{2bd}$, and $\dfrac{8ad^2}{9bc}$.

8. $\dfrac{3m^3}{4x^2}$, $\dfrac{2n^4}{21m^2}$, and $\dfrac{7x^2}{5mn^3}$.

9. $\dfrac{3x^2-x}{5}$ and $\dfrac{10}{2x^2-4x}$.

10. $\dfrac{x^2-16}{x^2+5x}$ and $\dfrac{x^2-25}{x^2-4x}$.

11. $\dfrac{a-b}{a^2+2ab}$ and $\dfrac{a^2-4b^2}{a^2-ab}$.

12. $\dfrac{a^2-2ab+b^2}{a+b}$ and $\dfrac{b}{ax-bx}$.

13. $\dfrac{xy+y^2}{x^3-y^3}$ and $\dfrac{x^3+x^2y+xy^2}{(x+y)^2}$.

14. $\dfrac{a^3-a^2+a}{x^2+2x+4}$ and $\dfrac{x^3-8}{a^3+1}$.

15. $1+\dfrac{4}{x}-\dfrac{5}{x^2}$ and $\dfrac{3x}{x^2+x-2}$.

16. $\dfrac{1-x^2}{1-y}$, $\dfrac{1-y^2}{x+x^2}$, and $\dfrac{1}{1-x}$.

17. $\dfrac{x^2+5xy+6y^2}{x^2-4xy-21y^2}$ and $\dfrac{x^2-7xy}{x^2-4y^2}$.

18. $\dfrac{x^2y-4y}{(x-y)^2-z^2}$ and $\dfrac{x^2-xy-xz}{xy+2y}$.

19. $\dfrac{x^3-y^3}{x^2-xy+y^2}$, $\dfrac{x^3+y^3}{x^2+xy+y^2}$, and $1+\dfrac{y}{x-y}$.

20. $\dfrac{a^2-(b-c)^2}{(a+c)^2-b^2}$ and $\dfrac{a^2-(b+c)^2}{(a-c)^2-b^2}$.

DIVISION OF FRACTIONS.

160. Required the quotient of $\dfrac{a}{b}$ divided by $\dfrac{c}{d}$.

By Art. 85, we are to find a quantity which, when multiplied by $\dfrac{c}{d}$, will produce $\dfrac{a}{b}$.

That quantity is evidently $\dfrac{ad}{bc}$; hence,

$$\frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}.$$

We observe that the quotient is obtained by multiplying the dividend, $\frac{a}{b}$, by $\frac{d}{c}$, which is the divisor inverted. We have then the following rule for the division of fractions:

Invert the divisor, and proceed as in multiplication.

Mixed quantities should be reduced to a fractional form before applying the rule.

If the divisor is an integer, it may be written in a fractional form, as explained in Art. 143.

EXAMPLES.

1. Divide $\frac{6a^2b}{5x^3y^4}$ by $\frac{9a^2b^3}{10x^2y^5}$.

By the rule,

$$\frac{6a^2b}{5x^3y^4} \div \frac{9a^2b^3}{10x^2y^5} = \frac{6a^2b}{5x^3y^4} \times \frac{10x^2y^5}{9a^2b^3} = \frac{4y}{3b^2x}, \text{ Ans}$$

2. Divide $\frac{x^2-9}{15}$ by $\frac{x^2+2x-3}{5}$.

$$\frac{x^2-9}{15} \div \frac{x^2+2x-3}{5} = \frac{(x+3)(x-3)}{15} \times \frac{5}{(x+3)(x-1)}$$

$$= \frac{x-3}{3(x-1)}, \text{ Ans}$$

Divide the following:

3. $\frac{7a^3b}{5m^2n^3}$ by $14ab^4$.

4. $\frac{18mx^3}{25ny^2}$ by $\frac{6m^2x^4}{5n^2y^5}$.

5. $\frac{1}{a^2+a-12}$ by $\frac{1}{a^2+3a-18}$.

6. $\frac{1}{4}-\frac{4}{x^2}$ by $\frac{x^2}{12}+\frac{x}{3}$.

7. $\dfrac{x^3-25x}{x^2+x-6}$ by $\dfrac{x^2-5x}{x^2-x-12}$.

8. $\dfrac{ab-b^2}{a^2+2ab+b^2}$ by $\dfrac{b^2}{a^2-b^2}$.

9. $\dfrac{m^3+n^3}{m^2-2mn+n^2}$ by $\dfrac{m^2+mn}{m-n}$.

10. $\dfrac{a+1}{a^2-3a}$ by $\dfrac{a^2-a-2}{a^2-a-6}$.

11. $9+\dfrac{5y^2}{x^2-y^2}$ by $3+\dfrac{5y}{x-y}$.

12. $\dfrac{a^4-8ab^3}{a^2-2ab-3b^2}$ by $\dfrac{a^3+2a^2b+4ab^2}{a-3b}$.

13. $\dfrac{2}{3y^2}-\dfrac{2}{xy}+\dfrac{3}{2x^2}$ by $\dfrac{2}{3y^2}-\dfrac{3}{2x^2}$.

COMPLEX FRACTIONS.

161. A **Complex Fraction** is one having a fraction in its numerator or denominator, or both.

It may be regarded as a case in division; its numerator answering to the dividend, and its denominator to the divisor.

EXAMPLES.

1. Reduce $\dfrac{a}{b-\dfrac{c}{d}}$ to its simplest form.

$$\frac{a}{b-\dfrac{c}{d}}=\frac{a}{\dfrac{bd-c}{d}}=a\times\frac{d}{bd-c}=\frac{ad}{bd-c},\ \textit{Ans.}$$

It is often advantageous to simplify a complex fraction by *multiplying both numerator and denominator by the lowest common multiple of their denominators.*

2. Reduce $\dfrac{\dfrac{a}{a-b}-\dfrac{a}{a+b}}{\dfrac{b}{a-b}+\dfrac{a}{a+b}}$ to its simplest form.

The L.C.M. of $a+b$ and $a-b$ is $(a+b)(a-b)$. Multiplying each of the component fractions by $(a+b)(a-b)$, we have

$$\frac{a(a+b)-a(a-b)}{b(a+b)+a(a-b)}=\frac{a^2+ab-a^2+ab}{ab+b^2+a^2-ab}=\frac{2ab}{a^2+b^2}, \textit{ Ans.}$$

3. Reduce $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}$ to its simplest form.

$$\frac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}=\frac{1}{1+\dfrac{x}{x+1}}=\frac{x+1}{x+1+x}=\frac{x+1}{2x+1}, \textit{ Ans.}$$

Reduce the following to their simplest forms:

4. $\dfrac{x-\dfrac{1}{x}}{1+\dfrac{1}{x}}$.

5. $\dfrac{\dfrac{1}{b}-\dfrac{1}{a}}{\dfrac{a}{b}-\dfrac{b}{a}}$.

6. $\dfrac{x^2+\dfrac{1}{x}}{1+\dfrac{1}{x}}$.

7. $\dfrac{a-2+\dfrac{1}{a}}{1-\dfrac{1}{a}}$.

8. $\dfrac{\dfrac{x^2+4y^2}{y}-4x}{\dfrac{1}{y}-\dfrac{2}{x}}$.

9. $\dfrac{x-7+\dfrac{12}{x}}{x+3-\dfrac{18}{x}}$.

10. $\dfrac{\dfrac{m^2}{n^3}+\dfrac{1}{m}}{\dfrac{m}{n^2}-\dfrac{m-n}{mn}}.$

11. $\dfrac{x-1-\dfrac{12}{x+3}}{x-5+\dfrac{12}{x+3}}.$

12. $2-\dfrac{1}{3+\dfrac{1}{\dfrac{x}{2}-2}}.$

13. $\dfrac{\dfrac{a}{b}-\dfrac{b^2}{a^2}}{\dfrac{a}{b}+1+\dfrac{b}{a}}.$

14. $\dfrac{\dfrac{x^2}{y^2}-4+\dfrac{3y^2}{x^2}}{\dfrac{x}{y}-\dfrac{3y}{x}}.$

15. $\dfrac{\dfrac{1}{1-x}-\dfrac{1}{1+x}}{\dfrac{1}{1-x}+\dfrac{1}{1+x}}.$

16. $\dfrac{1-\dfrac{2b-2c}{a+b-c}}{1+\dfrac{2c}{a-b-c}}.$

17. $\dfrac{1+\dfrac{2x^2}{1-x^2}}{1-x^2+\dfrac{4x^2}{1-x^2}}.$

18. $\dfrac{\dfrac{a+b}{c+d}+\dfrac{a-b}{c-d}}{\dfrac{a+b}{c-d}-\dfrac{a-b}{c+d}}.$

19. $\dfrac{1}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}.$

20. $\dfrac{\dfrac{x+2y}{x+y}+\dfrac{x}{y}}{\dfrac{x+2y}{y}-\dfrac{x}{x+y}}.$

21. $\dfrac{x-3a+\dfrac{4a^2}{a+x}}{x-\dfrac{2a^2}{a+x}}.$

22. $\dfrac{\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}.$

23. $\dfrac{\dfrac{m-n}{m+n}+\dfrac{m^2+n^2}{m^2-n^2}}{\dfrac{m^2}{m-n}+\dfrac{m^2n+n^3}{(m-n)^2}}.$

MISCELLANEOUS EXAMPLES.

162. Reduce the following to their simplest forms:

1. $c+\frac{2b}{x}-\frac{a+bx+cx^2}{x^2}$.

2. $\frac{m^3+4m^2-5m}{3m^3-75m}$.

3. $\frac{x^2(1+x)^3-x^3(1+x)^2}{(1+x)^6}$.

4. $\frac{a^2-b^2}{(m+n)^2}\div\frac{a^2-ab}{bm+bn}$.

5. $\frac{1+2x^2}{2+2x^2}-\frac{2+x}{2+2x}$.

6. $\frac{10a^2+30ab+20b^2}{5a^3+10a^2b}$.

7. $\left(x+1+\frac{1}{x}\right)\left(x-1+\frac{1}{x}\right)$.

8. $\frac{1-ax+a(x+a)}{(1-ax)^2+(x+a)^2}$.

9. $\left(\frac{a^2}{b^2}-2+\frac{b^2}{a^2}\right)\div\left(\frac{a}{b}-\frac{b}{a}\right)$.

10. $\frac{b(b-ax)+a(a+bx)}{(b-ax)^2+(a+bx)^2}$.

11. $\frac{ax}{ax+b}-\frac{b}{ax-b}+\frac{ax(3b-ax)}{a^2x^2-b^2}$.

12. $\frac{3x-3x^2}{2+4x+2x^2}\times\frac{10x+10x^2}{9-18x+9x^2}$.

13. $\frac{6n^5-48n^2}{9n^5+18n^4+36n^3}$.

14. $x^2-2y^2-\frac{6y^3-x^2y+xy^2}{x-3y}$.

15. $\frac{a+b}{a-b}-\frac{a-b}{a+b}-\frac{4b^2}{a^2-b^2}$.

16. $\left(x^3+x+\frac{1}{x}+\frac{1}{x^3}\right)\left(x-\frac{1}{x}\right)$.

17. $\frac{2x-1}{2x^2-2x+1}-\frac{2x+1}{2x^2+2x+1}$.

18. $\dfrac{\dfrac{1}{a}}{1+\dfrac{x^2}{a^2}}+\dfrac{1}{2}\left(\dfrac{1}{x-a}-\dfrac{1}{x+a}\right).$

19. $\dfrac{\dfrac{x}{x-y}-\dfrac{y}{x+y}}{\dfrac{x^2}{x^2+y^2}+\dfrac{y^2}{x^2-y^2}}.$ 20. $\dfrac{x^3-9x^2+26x-24}{x^3-12x^2+47x-60}.$

21. $a^2-3ab-2b^2-\dfrac{b^2(7a+6b)}{a-3b}.$

22. $\dfrac{2x+y}{x+y}-1-\dfrac{y}{y-x}-\dfrac{x^2}{x^2-y^2}.$

23. $\dfrac{(x+y+z)^2+(x-y)^2+(y-z)^2+(z-x)^2}{x^2+y^2+z^2}.$

24. $\dfrac{1}{x-2}-\dfrac{4}{(x-2)^2}-\dfrac{8(1-x)}{(x-2)^3}.$

25. $\dfrac{a^5-a^4b-ab^4+b^5}{a^4-a^3b-a^2b^2+ab^3}.$

26. $\dfrac{(4x+y)^2-(x-2y)^2}{(3x-4y)^2-(2x+3y)^2}.$

27. $\dfrac{1}{a+b}+\dfrac{1}{b+c}+\dfrac{1}{c+a}-\dfrac{(a+b+c)^2}{(a+b)(b+c)(c+a)}.$

28. $\dfrac{2(1-3x)}{(1+x)(1+9x)}-\dfrac{1-2x}{(1+x)(1+4x)}+\dfrac{2}{1+4x}.$

29. $\dfrac{1}{x-1}-\dfrac{1}{x+1}+\dfrac{3x^2}{x^3+1}-\dfrac{3x^2}{x^3-1}.$

30. $\left(\dfrac{a+b}{a-b}+\dfrac{a^2+b^2}{a^2-b^2}\right)\div\left(\dfrac{a-b}{a+b}-\dfrac{a^3-b^3}{a^3+b^3}\right).$

XII. SIMPLE EQUATIONS.

163. An **Equation** is a statement of the equality of two expressions.

The *First Member* of an equation is the expression on the left of the sign of equality, and the *Second Member* is the expression on the right of that sign.

Thus, in the equation $2x-3=3x-5$, the first member is $2x-3$, and the second is $3x-5$.

The *sides* of an equation are its two members.

164. A **Numerical Equation** is one in which all the known quantities are represented by numbers; as

$$2x-3=3x-5.$$

165. A **Literal Equation** is one in which some or all the known quantities are represented by letters; as

$$2x+3a=bx-4.$$

166. An **Identical Equation** is one whose two members are equal, whatever values are given to the letters involved; as

$$x^2-a^2=(x+a)(x-a).$$

167. The **Degree** of an equation, in which there is but one unknown quantity, is denoted by the highest power of the unknown quantity in the equation. Thus,

$$\left.\begin{array}{r} 2x-3=3x-5 \\ \text{and} \qquad a^2x=bc-d \end{array}\right\}\text{ are equations of the \textit{first degree}.}$$

$3x^2-2x=65$ is an equation of the *second degree*, etc.

168. A **Simple Equation** is an equation of the first degree.

169. The **Root** of an equation containing but one unknown quantity, is the value of the unknown quantity; or, it is the value which, when put in place of the unknown quantity, makes the equation identical.

Thus, the equation $5x - 7 = 3x + 1$, when 4 is put in place of x, becomes $20 - 7 = 12 + 1$, which is identical. Hence the root of the equation, or the value of x, is 4.

Note. A simple equation has but one root; but it will be seen hereafter that an equation may have two or more roots.

170. The *solution* of an equation is the process of finding its roots.

A root is *verified*, or the equation *satisfied*, when, on substituting the value of the root in place of its symbol, the equation becomes identical.

171. The operations required in the solution of an equation are based upon the following general principle, which is derived from the axioms of Art. 42:

If the same operations be performed upon equal quantities, the results will be equal.

Hence,

Both members of an equation may be increased, diminished, multiplied, or divided by the same quantity, without destroying the equality.

TRANSPOSITION.

172. *Any term may be transposed from one side of an equation to the other by changing its sign.*

For, consider the equation $x + a = b$.

Subtracting a from both members (Art. 171), we have

$$x + a - a = b - a;$$

or, by Art. 26, $$x = b - a,$$

where $+a$ has been transposed to the second member by changing its sign.

Again, consider the equation $x - a = b$.

Adding a to both members (Art. 171), we have

$$x - a + a = b + a;$$

or,

$$x = b + a.$$

where $-a$ has been transposed to the second member by changing its sign.

Note. If the same term appear in both members of an equation affected with the same sign, it may be suppressed.

173. *The signs of all the terms of an equation may be changed without destroying the equality.*

For, consider the equation $a - x = b - c$.

Transposing each term (Art. 172), we have

$$c - b = x - a;$$

or,

$$x - a = c - b,$$

which is the same as the original equation with every sign changed.

SOLUTION OF SIMPLE EQUATIONS.

174. **1.** Solve the equation $5x - 7 = 3x + 1$.

Transposing the unknown quantities to the first member, and the known quantities to the second, we have

$$5x - 3x = 7 + 1.$$

Uniting the similar terms, $2x = 8$.

Dividing both members by 2 (Art. 171),

$$x = 4, \textit{ Ans.}$$

Note. The result may be verified by substituting the value of x in the given equation, as shown in Art. 169.

We have then the following rule for the solution of a simple equation containing but one unknown quantity:

Transpose the unknown terms to the first member, and the known terms to the second.

Unite the similar terms, and divide both members by the coefficient of the unknown quantity.

EXAMPLES.

2. Solve the equation $14 - 5x = 19 + 3x$.

Transposing, $-5x - 3x = 19 - 14$.

Uniting terms, $-8x = 5$.

Dividing by -8, $x = -\frac{5}{8}$, *Ans.*

Note. To verify this result, put $x = -\frac{5}{8}$ in the given equation. Then,

$$14 - 5\left(-\frac{5}{8}\right) = 19 + 3\left(-\frac{5}{8}\right)$$

Or, $$14 + \frac{25}{8} = 19 - \frac{15}{8}$$

Or, $$\frac{137}{8} = \frac{137}{8};$$ which is identical.

Solve the following equations:

3. $8x = 5x + 42$.

4. $7x + 5 = -30$.

5. $7x + 5 = x + 23$.

6. $9x + 7 = 3x - 11$.

7. $3x - 8 = 5x + 8$.

8. $5 - 6x = 1 - 4x$.

9. $5x + 14 = 17 - 3x$.

10. $3x - 31 = 11x - 16$.

11. $18 - 7x = 18x - 7$.

12. $27 + 10x = 13x + 23$.

13. $19x - 11 = 15 + 6x$.

14. $32x - 15 = 7 + 65x$.

15. $13x - 81 = 5x - 31x - 159.$

16. $12x - 20x + 13 = 9x - 259.$

17. Solve the equation

$$(2x-3)^2 - x(x+1) = 3(x-2)(x+7) - 5.$$

Performing the operations indicated, we have

$$4x^2 - 12x + 9 - x^2 - x = 3x^2 + 15x - 42 - 5.$$

Transposing,

$$4x^2 - 12x - x^2 - x - 3x^2 - 15x = -42 - 5 - 9.$$

Uniting terms, $\quad -28x = -56.$

Dividing by -28, $\quad x = 2$, *Ans.*

Solve the following equations:

18. $3 + 2(2x+3) = 2x - 3(2x+1).$

19. $2x - (4x-1) = 5x - (x-1).$

20. $7(x-2) - 5(x+3) = 3(2x-5) - 6(4x-1).$

21. $3(3x+5) - 2(5x-3) = 13 - (5x-16).$

22. $(2x-1)(3x+2) = (3x-5)(2x+20).$

23. $(5-6x)(2x-1) = (3x+3)(13-4x).$

24. $(x-3)^2 - (5-x)^2 = -4x.$

25. $(2x-1)^2 - 3(x-2) + 5(3x-2) - (5-2x)^2 = 0.$

26. $2(x-2)^2 - 3(x-1)^2 + x^2 = 1.$

27. $(x-1)(x-2)(x+4) = (x+2)(x+3)(x-4).$

28. $5(7+3x) - (2x-3)(1-2x) - (2x-3)^2 - (5+x) = 0.$

29. $(5x-1)^2 - (3x+2)^2 - (4x-3)^2 + 4 = 0.$

30. $(2x+1)^3 + (2x-1)^3 = 16x(x^2-4) - 228.$

SOLUTION OF EQUATIONS CONTAINING FRACTIONS.

175. 1. Solve the equation $\frac{2x}{3}-\frac{5}{4}=\frac{5x}{6}-\frac{9}{8}$.

The L.C.M. of 3, 4, 6, and 8 is 24. Multiplying each term of the equation by 24, we have

$$16x-30 \quad = 20x - 27$$

$$16x - 20x = 30 \quad - 27$$

$$-4x = 3$$

$$x = -\frac{3}{4}, \textit{ Ans.}$$

We have then the following rule for clearing an equation of fractions:

Multiply each term by the lowest common multiple of the denominators.

EXAMPLES.

Solve the following equations:

2. $x+\frac{x}{2}+\frac{x}{3}=-11$.

3. $\frac{3x}{4}-\frac{5x}{6}+\frac{1}{18}=0$.

4. $2x-\frac{3x}{4}=\frac{13}{14}-\frac{x}{7}$.

5. $\frac{7x}{4}-7=\frac{5x}{3}-\frac{9x}{4}$.

6. $\frac{1}{6}+\frac{1}{2x}=\frac{1}{4}+\frac{1}{12x}$.

7. $\frac{2x}{5}-\frac{9x}{20}-\frac{7x}{10}=\frac{5}{4}$.

8. $\frac{3}{x}-\frac{5}{2x}=7-\frac{3}{2x}$.

9. $\frac{x}{2}+\frac{11}{6}-\frac{x}{3}=\frac{x}{6}-\frac{3x}{4}$.

10. $x-\frac{x}{7}+20=\frac{x}{2}+\frac{x}{4}+26$.

11. $\frac{3}{x}-\frac{7}{2x}=\frac{7}{12}-\frac{5}{3x}$.

12. Solve the equation $\frac{3x-1}{4}-\frac{4x-5}{5}=4+\frac{7x+5}{10}$.

Multiplying through by 20, the L.C.M. of 4, 5, and 10,

$$15x-5-(16x-20)=80+14x+10$$
$$15x-5-16x+20=80+14x+10$$
$$15x-16x-14x=80+10+5-20$$
$$-15x=75$$
$$x=-5, \textit{ Ans.}$$

Note. If a fraction whose numerator is a polynomial is preceded by a $-$ sign, care must be taken to change the sign of each term of the numerator when the denominator is removed. It is convenient, in such a case, to enclose the numerator in a parenthesis, as shown in the above example.

13. $3x+\frac{5x+3}{7}=\frac{7x}{2}$.

14. $x-\frac{2x+1}{5}=5x-\frac{5}{3}$.

15. $7x-\frac{11x-3}{4}=3x+7$.

16. $4x-\frac{2x-3}{3}+\frac{1}{2}(x-9)=5x$.

17. $x-(3x-4)-\frac{5-2x}{4}=2$.

18. $\frac{2x}{21}=x-7+\frac{x+3}{15}$.

19. $\frac{x+1}{2}-\frac{x+4}{5}=\frac{x-4}{7}$.

20. $2-\frac{7x-1}{6}=3x-\frac{19x+3}{4}$.

21. $\frac{5x-2}{3}-\frac{3x+4}{4}-\frac{7x+2}{6}=\frac{x-10}{2}$.

22. $\frac{1}{2}(x+1)-\frac{2x-5}{5}=\frac{11x+5}{10}-\frac{x-13}{3}$.

23. $\dfrac{5x+1}{3}+\dfrac{17x+7}{9}-\dfrac{1}{2}(3x-1)=\dfrac{7x-1}{6}.$

24. $\dfrac{4+x}{7}=\dfrac{1}{2}(3x-2)-\dfrac{11x+2}{14}-\dfrac{1}{3}(2-9x).$

25. $\dfrac{2x+1}{3}=\dfrac{4x+5}{4}-\dfrac{8+x}{6}+\dfrac{2x+5}{8}.$

26. $\dfrac{5x-1}{2}-\dfrac{7-3x}{3x}=\dfrac{10x-3}{4}-\dfrac{3-5x}{2x}.$

27. $\dfrac{3x+7}{2}-\dfrac{4(x^2-2)}{3x}-\dfrac{x^3+16}{6x^2}=\dfrac{7}{2}.$

28. Solve the equation $\dfrac{2}{x-1}-\dfrac{3}{x+1}-\dfrac{1}{x^2-1}=0.$

Multiplying each term by x^2-1, the L.C.M. of the denominators,

$$2(x+1)-3(x-1)-1=0$$
$$2x+2-3x+3-1=0$$
$$2x-3x=-2-3+1$$
$$-x=-4$$
$$x=4, \textit{ Ans.}$$

29. Solve the equation $\dfrac{6x+1}{15}-\dfrac{2x-4}{7x-16}=\dfrac{2x-1}{5}.$

Multiplying each term by 15,

$$6x+1-\frac{30x-60}{7x-16}=6x-3.$$

Transposing and uniting terms, $4=\dfrac{30x-60}{7x-16}.$

Multiplying by $7x-16$, $28x-64=30x-60$

$$-2x=4$$
$$x=-2, \textit{ Ans.}$$

Note. If the denominators are partly monomial and partly polynomial, it is often advantageous to clear of fractions at first partially; multiplying by a quantity which will remove the *monomial* denominators.

Solve the following equations:

30. $\dfrac{1}{3x-7}-\dfrac{2}{3x+7}=0.$

31. $\dfrac{2x-1}{3x+4}=\dfrac{2x+7}{3x+2}.$

32. $\dfrac{6x^2-7x+5}{2x^2+5x-13}=3.$

33. $\dfrac{5x-2}{x(x-1)}=\dfrac{5x+7}{x^2-1}.$

34. $\dfrac{x}{3}-\dfrac{x^2-5x}{3x-7}=\dfrac{2}{3}.$

35. $\dfrac{(x+5)^2}{x-3}=\dfrac{5x+1}{5}.$

36. $\dfrac{1}{x+1}+\dfrac{2}{x+2}=\dfrac{3}{x+3}.$

37. $\dfrac{3x+2}{6}-\dfrac{2x-1}{3x-7}=\dfrac{x}{2}.$

38. $\dfrac{2}{x-2}-\dfrac{1}{x-3}=\dfrac{1}{x^2-5x+6}.$

39. $\dfrac{6x+7}{15}-\dfrac{2(x-1)}{7x-6}=\dfrac{2x+1}{5}.$

40. $\dfrac{3}{1-x}-\dfrac{2}{1+x}-\dfrac{1}{1-x^2}=0.$

41. $\dfrac{2x^2+3x}{2x+1}+\dfrac{1}{3x}=x+1.$

42. $2\left(\dfrac{x+1}{x+2}\right)+3\left(\dfrac{x+2}{x+1}\right)=5.$

43. $\dfrac{6}{3x+1}-\dfrac{1}{x+1}=\dfrac{2}{2x-1}.$

44. $\dfrac{x}{9}=\dfrac{x+1}{3}-\dfrac{7-2x^2}{1-9x}.$

45. $\dfrac{(x+1)^2}{(x+2)^2}=\dfrac{x-4}{x-2}.$

46. $\dfrac{2x^2-3x+2}{3x^2+x-1}=\dfrac{2x-3}{3x+1}$.

47. $\dfrac{x-1}{x-2}+\dfrac{x+1}{x+2}=\dfrac{2(x^2+4x+1)}{(x+2)^2}$.

48. $\dfrac{4x+3}{10}-\dfrac{12x-5}{5x-29}-\dfrac{2x-1}{5}=0$.

49. $\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}=\dfrac{x-3}{x-4}-\dfrac{x-4}{x-5}$.

SOLUTION OF LITERAL EQUATIONS.

176. 1. Solve the equation $2ax-3b=x+c-3ax$.

Transposing and uniting terms, $5ax-x=3b+c$.

Factoring the first member, $x(5a-1)=3b+c$.

Dividing by $5a-1$, $x=\dfrac{3b+c}{5a-1}$, *Ans.*

2. Solve the equation $(b-cx)^2-(a-cx)^2=b(b-a)$.

Performing the operations indicated,

$$b^2-2bcx+c^2x^2-(a^2-2acx+c^2x^2)=b^2-ab$$

$$b^2-2bcx+c^2x^2-a^2+2acx-c^2x^2=b^2-ab$$

$$2acx-2bcx=a^2-ab$$

Factoring both members, $2cx(a-b)=a(a-b)$

Dividing by $2c(a-b)$, $x=\dfrac{a(a-b)}{2c(a-b)}$

$$=\frac{a}{2c}, \text{ Ans.}$$

EXAMPLES.

Solve the following equations:

3. $2ax + d = 3c - bx.$

4. $6bmx - 5an = 15am - 2bnx.$

5. $x + 1 = 2ax - a^2(x - 1).$

6. $\frac{a^2}{x} + \frac{b}{2} = \frac{4b^2}{x} + \frac{a}{4}.$

7. $(a^2 - 2x)^2 = (4x - 3a^2)(x + a^2).$

8. $(2m + 3x)(2m - 3x) = n^2 - (3x - n)^2.$

9. $\frac{x - a}{b} - \frac{x + b}{a} + 2 = 0.$

10. $(x - a - b)^2 - (x - a)(x - b) + ab = 0.$

11. $\frac{x}{x - a} - \frac{x + 2b}{x + a} = \frac{a^2 + b^2}{x^2 - a^2}.$

12. $\frac{(b - 3x)(c + 2x)}{2(x - c)(b - 3c - 3x)} = 1.$

13. $(x+a)^3 - (x-a)^3 - a(3x-a)(2x+a) = x(a+1) + 3.$

14. $\frac{(n^2 - x^2)(n + x)}{x + 2n} = -x^2 + nx + n^2.$

15. $(a - x)(b - x) - a(b + 1) = \frac{a^2}{b} + x^2.$

16. $\frac{x}{2a} - 3 + \frac{x}{4a^3} = \frac{x}{3a^2} - 2a(2 - 3a).$

17. $\frac{x}{2} + \frac{1 - 2ax}{2a} + \frac{2x - 1}{a^2} = 0.$

18. $\dfrac{x}{mn}-\dfrac{x+mn}{3n}=\dfrac{x}{3n}-(m-1)$.

19. $\dfrac{x+2a}{x-a}+\dfrac{x-3a}{x+a}=2$.

20. $\dfrac{4x-a}{2x-a}-\dfrac{x+a}{x-a}=1$.

21. $\dfrac{x}{2}-\dfrac{a-bcx}{2bc}=\dfrac{x}{6c}-\dfrac{ac-4bx}{3bc}$.

22. $\dfrac{ax+b}{ax-b}-\dfrac{3b}{ax+b}=\dfrac{a^2x^2+b^2}{a^2x^2-b^2}$.

23. $\dfrac{ax-b}{ax+b}-\dfrac{bx-a}{bx+a}=\dfrac{a-b}{(ax+b)(bx+a)}$.

24. $\dfrac{x-n}{m}-\dfrac{x^2-mx-n^2}{mx-n^2}=1+\dfrac{n^2}{mx-n^2}$.

SOLUTION OF EQUATIONS INVOLVING DECIMALS.

177. 1. Solve the equation $.2x-.01-.03x=.113x+.161$.

Changing the decimals into common fractions,

$$\frac{2x}{10}-\frac{1}{100}-\frac{3x}{100}=\frac{113x}{1000}+\frac{161}{1000}.$$

Clearing of fractions,

$$200x-10-30x=113x+161$$

$$57x=171$$

$$x=3, \textit{ Ans.}$$

Or, we may solve the equation as follows:

Transposing, $.2x-.03x-.113x=.01+.161.$

Uniting terms, $.057x=.171.$

Dividing by .057, $x=3$, *Ans.*

EXAMPLES.

Solve the following equations:

2. $.23x-2.05=.02x-1.882.$

3. $.001x-.32=.09x-.2x-.653.$

4. $.3x-.02-.003x=.7-.06x-.006.$

5. $.3(1.2x-5)=14+.05x.$

6. $.7(x+.13)=.03(4x-.1)+.5.$

7. $3.3x-\frac{.72x-.55}{.5}=.1x+9.9.$

8. $4.25-\frac{.2}{x}=\frac{17}{4}-\frac{1-.1x}{x}.$

9. $\frac{.6x+.044}{.4}-\frac{.5x-.178}{.6}=.38.$

10. $\frac{2-3x}{1.5}+\frac{5x}{1.25}-\frac{2x-3}{9}=\frac{x-2}{1.8}+\frac{25}{9}.$

XIII. PROBLEMS.

LEADING TO SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

178. For the solution of a problem by Algebra no general rule can be given, as much must depend on the skill and ingenuity of the student. A few suggestions, however, may be found of service:

1. *Express the unknown quantity, or one of the unknown quantities, by one of the final letters of the alphabet.*

2. *From the given conditions, find expressions for the other unknown quantities, if any, in the problem.*

3. *Form an equation in accordance with the conditions of the problem.*

4. *Solve the equation thus formed.*

PROBLEMS.

179. **1.** What number is that to which if four-sevenths of itself be added, the sum will equal twice the number diminished by 27?

Let $x =$ the number.

Then, $\frac{4x}{7} =$ four-sevenths of it,

and $2x =$ twice it.

By the conditions, $x + \frac{4x}{7} = 2x - 27$

$$7x + 4x = 14x - 189$$

$$-3x = -189.$$

Whence, $x = 63$, the number required.

2. A is three times as old as B, and eight years ago he was seven times as old as B. Required their ages at present.

Let $x = \text{B's age.}$

Then, $3x = \text{A's age.}$

Also, $x - 8 = \text{B's age 8 years ago,}$

and $3x - 8 = \text{A's age 8 years ago.}$

By the conditions, $3x - 8 = 7(x - 8)$

$$3x - 8 = 7x - 56$$

$$-4x = -48.$$

Whence, $x = 12$, B's age,

and $3x = 36$, A's age.

Note. In the above solution we say "Let $x =$ B's age," meaning "Let $x =$ the number of years in B's age." Abbreviations of this nature are often used in Algebra; but it should be remembered that they are in fact abbreviations, and that x can only represent an abstract number.

3. A had twice as much money as B; but, after giving B \$35, he had only one-third as much as B. How much had each at first?

Let $x =$ what B had at first.

Then, $2x =$ what A had at first.

After giving B \$35, A had left $2x - 35$ dollars, while B had $x + 35$ dollars. Then, by the conditions,

$$x + 35 = 3(2x - 35)$$

$$x + 35 = 6x - 105$$

$$-5x = -140.$$

Whence, $x = 28$, B's money at first,

and $2x = 56$, A's money at first.

4. What number is that whose double exceeds its half by 45?

5. Divide 34 into two parts such that four-sevenths of one part may be equal to two-fifths of the other.

6. What number exceeds the sum of its third, tenth, and twelfth parts by 58?

7. Divide 59 into two parts such that the sum of one-seventh the greater and one-third the less shall be equal to 13.

8. A is four times as old as B, and in 30 years he will be only twice as old as B. What are their ages?

9. A is 62 years of age, and B is 36. How many years is it since A was three times as old as B?

10. A had one-half as much money as B; but after B had given him \$42, he had four times as much as B. How much had each at first?

11. Divide 207 into two parts such that one-fourth the greater shall exceed two-sevenths the less by 3.

12. What two numbers are those whose difference is 3, and the difference of whose squares is 51?

13. A drover paid \$1428 for a lot of oxen and cows. For the oxen he paid \$55 each, and for the cows \$32 each; and he has twice as many cows as oxen. How many has he of each?

14. Divide 80 into two parts such that if the greater is taken from 62, and the less from 48, the remainders are equal.

15. A gentleman left an estate of \$1872 to be divided between his wife, three sons, and two daughters. The wife was to receive three times as much as either of the daughters, and each son one-half as much as each of the daughters. How much did each receive?

16. Divide \$70 between A, B, and C, so that A's share may be three-eighths of B's, and C's share two-ninths of A's.

17. In a garrison of 2744 men, there are $12\frac{1}{2}$ times as many infantry as cavalry, and twice as many cavalry as artillery. How many are there of each kind?

18. A is 34 years older than B; and he is as much above 50 as B is below 40. Required their ages.

19. A man travelled 3036 miles. He went four-sevenths as many miles on foot as by water, and two-fifths as many miles on horseback as by water. How many miles did he travel in each manner?

20. Divide a into two parts such that m times the first part shall be equal to n times the second.

Let $x =$ the first part.

Then, $a - x =$ the second part.

By the conditions, $mx = n(a - x)$.

Or, $mx + nx = an$.

Whence, $x = \frac{an}{m+n}$, the first part.

Therefore, $a - x = a - \frac{an}{m+n} = \frac{am}{m+n}$, the second part.

21. Divide a into two parts such that m times the first shall be equal to the second divided by n.

22. Find four consecutive numbers whose sum is 94.

23. Divide 43 into two parts such that one of them shall be three times as much above 20 as the other lacks of 17.

24. Divide \$47 between A, B, C, and D, so that A and B together may have \$27, A and C \$25, and A and D \$23.

25. If a certain number is increased by 15, one-half the result is as much below 80 as the number itself is above 100. Required the number.

26. Divide 205 into four parts such that the second is one-half of the first, the third one-third of the second, and the fourth one-fourth of the third.

27. Eleven years ago, A was 4 times as old as B, and in 13 years he will be only twice as old. Required their ages at present.

28. Find two consecutive numbers such that the difference of their squares added to three times the greater number exceeds the less number by 92.

29. What number is that, five-sixths of which as much exceeds 25 as one-ninth of it is below 9?

30. A is m times as old as B, and in a years he will be n times as old. Required their ages at present.

31. Divide a into three parts such that the first may be n times the second, and the second n times the third.

32. A can do a piece of work in 8 days which B can perform in 10 days. In how many days can it be done by both working together?

Let $x =$ the number of days required.

Then, $\frac{1}{x} =$ what both can do in one day.

Also, $\frac{1}{8} =$ what A can do in one day,

and $\frac{1}{10} =$ what B can do in one day.

By the conditions, $\frac{1}{8} + \frac{1}{10} = \frac{1}{x}.$

$$5x + 4x = 40$$

$$9x = 40.$$

Whence, $x = 4\frac{4}{9}$, the number of days required.

33. A can do a piece of work in 15 days, and B can do the same in 18 days. In how many days can it be done by both working together?

34. A can do a piece of work in $3\frac{2}{3}$ hours which B can do in $2\frac{3}{4}$ hours, and C in $2\frac{1}{5}$ hours. In how many hours can it be done by all working together?

35. The stones which pave a square court would just cover a rectangular area whose length is 6 yards longer, and breadth 4 yards shorter, than the side of the square. Required the area of the court.

36. A, B, and C found a sum of money. It was agreed that A should receive \$15 less than one-half, B \$13 more than one-fourth, and C the remainder, which was \$27. How much did A and B receive?

37. A can do a piece of work in a hours which B can do in b hours. In how many hours can it be done by both working together?

38. A vessel can be filled by three taps; by the first alone it can be filled in a minutes, by the second in b minutes, and by the third in c minutes. In what time will it be filled if all the taps are opened?

39. A sum of money, amounting to $4.32, consists entirely of dimes and cents, there being in all 108 coins. How many are there of each kind?

Let $x =$ the number of dimes.

Then, $108 - x =$ the number of cents.

Also, $10x =$ the value of the dimes in *cents*.

By the conditions,

$$10x + 108 - x = 432$$

$$9x = 324.$$

Whence, $x = 36$, the number of dimes,

and $108 - x = 72$, the number of cents.

40. A man has $4.04 in dollars, dimes, and cents. He has one-fifth as many cents as dimes, and twice as many cents as dollars. How many has he of each kind?

41. A man has 3 shillings 7 pence in two-penny pieces and farthings; and he has 19 more farthings than two-penny pieces. How many has he of each kind?

42. I bought a picture for a certain sum, and paid the same price for a frame. If the frame had cost $1 less, and the picture 75 cents more, the price of the frame would have been only half that of the picture. Required the cost of the picture.

43. A laborer agreed to serve for 36 days on condition that for every day he worked he should receive $1.25, and for every day he was absent he should forfeit 50 cents. At the end of the time he received $17. How many days did he work, and how many was he absent?

44. A has \$105, and B \$83. After giving B a certain sum, A has only one-third as much money as B. How much was given to B?

45. A has a dollars, and B b dollars. After giving B a certain sum, A has c times as much money as B. How much was given to B?

46. A vessel can be emptied by three taps; by the first alone it can be emptied in 80 minutes, by the second in 200 minutes, and by the third in 5 hours. In what time will it be emptied if all the taps are opened?

47. The second digit of a number exceeds the first by 2; and if the number, increased by 6, be divided by the sum of its digits, the quotient is 5. Required the number.

Let $x =$ the first digit.

Then, $x + 2 =$ the second,

and $2x + 2 =$ the sum of the digits.

The number itself is equal to 10 times the first digit, plus the second, which is $10x + x + 2$, or $11x + 2$. Hence, by the conditions,

$$\frac{11x + 2 + 6}{2x + 2} = 5$$

$$11x + 8 = 10x + 10.$$

Whence, $x = 2$.

Therefore, $11x + 2 = 24$, the number required.

48. The first digit of a number exceeds the second by 4; and if the number be divided by the sum of its digits, the quotient is 7. Required the number.

49. The first digit of a number is three times the second; and if the number, increased by 3, be divided by the difference of its digits, the quotient is 16. Required the number.

50. A merchant has grain worth 9 shillings per bushel, and other grain worth 13 shillings per bushel. In what proportion must he mix 40 bushels, so that the mixture may be worth 10 shillings per bushel?

51. Gold is $19\frac{1}{4}$ times as heavy as water, and silver $10\frac{1}{2}$ times. A mixed mass weighs 4160 ounces, and displaces 250 ounces of water. How many ounces of each metal does it contain?

52. The second digit of a number exceeds the first by 3; and if the number, diminished by 9, be divided by the sum of its digits, the quotient is 3. Required the number.

53. Two persons, A and B, 63 miles apart, start at the same time and travel towards each other. A travels 4 miles an hour, and B 3 miles an hour. How far will each have travelled when they meet?

Let $x =$ the distance A travels.

Then, $63 - x =$ the distance B travels.

Also, $\frac{x}{4} =$ the time A takes to travel x miles,

and $\frac{63 - x}{3} =$ the time B takes to travel x miles.

By the conditions, $$\frac{x}{4} = \frac{63 - x}{3}$$

$$3x = 252 - 4x$$

$$7x = 252.$$

Whence, $x = 36$, the distance A travels,

and $63 - x = 27$, the distance B travels.

54. A person has $4\frac{1}{4}$ hours at his disposal. How far can he ride in a coach which travels 5 miles an hour, so as to return home in time, walking back at the rate of $3\frac{1}{2}$ miles an hour?

55. A courier who travels a miles daily is followed after n days by another, who travels b miles daily. In how many days will the second overtake the first?

56. Two men, A and B, 26 miles apart, set out, B 30 minutes after A, and travel towards each other. A travels 3 miles an hour, and B 4 miles an hour. How far will each have travelled when they meet?

57. A capitalist invests $\frac{3}{8}$ of a certain sum in 5 per cent bonds, and the remainder in 6 per cent bonds; and finds that his annual income is \$180. Required the amount in each kind of bond.

58. What principal at r per cent interest will amount to a dollars in t years?

59. In how many years will p dollars amount to a dollars, at r per cent interest?

60. Separate 41 into two parts such that one divided by the other may give 1 as a quotient and 5 as a remainder.

Let $x =$ the divisor.

Then, $41 - x =$ the dividend.

By the conditions, $\dfrac{41 - x}{x} = 1 + \dfrac{5}{x}$

$$41 - x = x + 5$$

$$-2x = -36.$$

Whence, $x = 18$, the divisor,

and $41 - x = 23$, the dividend.

61. Separate 37 into two parts such that one divided by the other may give 3 as a quotient and 1 as a remainder.

62. Separate 113 into two parts such that one divided by the other may give 2 as a quotient and 20 as a remainder.

63. A general, arranging his men in a solid square, finds he has 21 men over. But attempting to add 1 man to each side of the square, he finds he wants 200 men to fill up the square. Required the number of men on a side at first, and the whole number of troops.

64. Separate a into two parts such that one divided by the other may give b as a quotient and c as a remainder.

65. The denominator of a fraction exceeds the numerator by 6; and if 8 is added to the denominator, the value of the fraction is $\frac{1}{3}$. Required the fraction.

66. The sum of the digits of a number is 6, and the number exceeds its first digit by 46. What is the number?

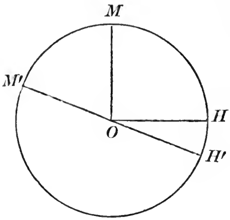
67. At what rate of interest will p dollars amount to a dollars in t years?

68. A man bought a picture for a certain price, and paid three-fourths the same amount for a frame. If the frame had cost \$2 less, and the picture 60 cents more, the price of the frame would have been one-third that of the picture. How much did each cost?

69. The denominator of a fraction exceeds the numerator by 1. If the denominator be increased by 2, the resulting fraction is less by unity than twice the original fraction. Required the fraction.

70. At what time between 3 and 4 o'clock are the hands of a watch opposite to each other?

Let OM and OH represent the positions of the minute and hour-hands at 3 o'clock, and OM' and OH' their positions when opposite to each other.



Let $x=$ the arc $MHH'M'$ over which the minute-hand has passed since 3 o'clock.

Then, $\frac{x}{12}=$ the arc HH' over which the hour hand has passed since 3 o'clock.

Also, the arc $MH = 15$ minute-spaces,
and the arc $H'M' = 30$ minute-spaces.

Now, arc $MHH'M' =$ arc $MH +$ arc $H'M' +$ arc HH'.

That is, $$x = 15 + 30 + \frac{x}{12}$$

$$12x = 540 + x$$

$$11x = 540.$$

Whence, $$x = 49\tfrac{1}{11}.$$

Hence the required time is $49\frac{1}{11}$ minutes after 3 o'clock.

71. At what time between 7 and 8 are the hands of a watch opposite to each other?

72. At what time between 2 and 3 are the hands of a watch opposite to each other?

73. At what time between 5 and 6 are the hands of a watch together?

74. At what time between 1 and 2 are the hands of a watch together?

75. A woman sells half an egg more than half her eggs. Again she sells half an egg more than half her remaining eggs. A third time she does the same; and now she has sold all her eggs. How many had she at first?

76. A man has two kinds of money, dimes and half-dimes. If he is offered $1.35 for 20 coins, how many of each kind must he give?

77. A man has a hours at his disposal. How far can he ride in a coach which travels b miles an hour, so as to return home in time, walking back at the rate of c miles an hour?

78. At what time between 6 and 6.30 o'clock are the hands of a watch at right angles to each other?

79. At what times between 10 and 11 o'clock are the hands of a watch at right angles?

80. A banker has two kinds of money. It takes a pieces of the first kind to make a dollar, and b pieces of the second kind. If he is offered a dollar for c pieces, how many of each kind must he give?

81. A alone can perform a piece of work in 12 hours; A and C together can do it in 5 hours; and C's work is two-thirds of B's. The work must be completed at noon. A commences work at 5 A.M.; at what hour can he be relieved by B and C, and the work be just finished in time?

82. At what time between 4 and 5 is the minute-hand of a watch exactly 5 minutes in advance of the hour-hand?

83. A man buys a certain number of eggs at the rate of 3 for 10 cents. He sells one-third of them at the rate of 2 for 7 cents, and the remainder at the rate of 4 for 15 cents; and makes 16 cents by the transaction. How many eggs did he buy?

84. A merchant increases his capital annually by one-third of it, and at the end of each year sets aside \$2700 for expenses. At the end of four years, after deducting the amount for expenses, he finds that his capital is reduced to \$2980. What was his capital at first?

85. A man owns a harness valued at \$25, a horse, and a carriage. The harness and carriage are together worth two-thirds the value of the horse, and the horse and harness are together worth \$15 more than twice the value of the carriage. Required the value of the horse, and of the carriage.

86. Two men, A and B, 107 miles apart, set out at the same time and travel towards each other. A travels at the rate of 13 miles in 5 hours, and B at the rate of 11 miles in 4 hours. How far will each have travelled when they meet?

87. A mixture is made of a pounds of coffee at m cents a pound, b pounds at n cents, and c pounds at p cents. Required the cost per pound of the mixture.

88. A, B, and C together can do a piece of work in 6 days; B's work is one-half of A's, and C's work is two-thirds of B's. How many days will it take each working alone?

89. A and B start in business, A putting in $\frac{3}{2}$ as much capital as B. The first year, A gains \$150, and B loses $\frac{1}{4}$ of his money. The next year, A loses $\frac{1}{4}$ of his money, and B gains \$300; and they have now equal amounts. How much had each at first?

90. At what time between 9 and 10 is the hour-hand of a watch exactly one minute in advance of the minute-hand?

91. A and B together can do a piece of work in $1\frac{5}{7}$ days, A and C in $1\frac{7}{8}$ days, and B and C in $2\frac{2}{9}$ days. How many days will it take each working alone?

92. A man buys two pieces of cloth, one of which contains 3 yards more than the other. For the larger piece he pays at the rate of \$5 for 6 yards, and for the other at the rate of \$7 for 5 yards. He sells the whole at the rate of 3 yards for \$4, and makes \$8 by the transaction. How many yards were there in each piece?

93. A gentleman distributing some money among beggars, found that in order to give them a cents each, he should need b cents more. He therefore gave them c cents each, and had d cents left. Required the number of beggars.

94. A man let a certain sum for 3 years at 5 per cent compound interest; that is, at the end of each year there was added $\frac{1}{20}$ to the sum due. At the end of the third year there was due him \$2315.25. Required the sum let.

95. A man starts in business with \$4000, and adds to his capital annually one-fourth of it. At the end of each year he sets aside a fixed sum for expenses. At the end of three years, after deducting the fixed sum for expenses, his capital is reduced to \$2475. What are his annual expenses?

96. A man invests one-third of his money in $3\frac{1}{2}$ per cent bonds, two-fifths in 4 per cent bonds, and the balance in $4\frac{1}{2}$ per cent bonds. His income from the investments is \$595. What is the amount of his property?

97. At what time between 8 and 9 o'clock is the minute-hand of a watch exactly 35 minutes in advance of the hour-hand?

98. A fox is pursued by a greyhound, and has a start of 60 of her own leaps. The fox makes 3 leaps while the greyhound makes but 2; but the latter in 3 leaps goes as far as the former in 7. How many leaps does each make before the greyhound catches the fox?

XIV. SIMPLE EQUATIONS.

CONTAINING TWO UNKNOWN QUANTITIES.

180. If we have a simple equation containing *two* unknown quantities, as $x + y = 12$, it is impossible to determine the values of x and y *definitely;* because, if any value be assumed for one of the quantities, we can find a corresponding value for the other.

Thus, if $x = 9$, then $9 + y = 12$, or $y = 3$;

if $x = 8$, then $8 + y = 12$, or $y = 4$; etc.

Hence, any of the pairs of values,

$$x = 9,\ y = 3;\ x = 8,\ y = 4;\ \text{etc.},$$

will satisfy the given equation.

Similarly, the equation $x - y = 4$ is satisfied by any of the following pairs of values:

$$x = 9,\ y = 5;\ x = 8,\ y = 4;\ \text{etc.}$$

Equations of this kind are called *indeterminate.*

But suppose we are required to find a pair of values which will satisfy both $x + y = 12$ and $x - y = 4$ at the same time. It is evident by inspection that the values

$$x = 8,\ y = 4$$

satisfy both equations; and no other pair of values can be found which will satisfy both simultaneously.

181. Simultaneous Equations are such as are satisfied by the *same values* of their unknown quantities.

Independent Equations are such as cannot be made to assume the same form.

Thus, $x+y=9$ and $x-y=1$ are independent equations.

But $x+y=9$ and $2x+2y=18$ are not independent, since the first equation may be obtained from the second by dividing each term by 2.

182. It is evident from Art. 180 that two unknown quantities require for their determination *two* independent, simultaneous equations.

Two such equations may be solved by combining them so as to form a single equation containing but *one* unknown quantity. This operation is called **Elimination.**

183. There are three principal methods of elimination:

1. By Addition or Subtraction.
2. By Substitution.
3. By Comparison.

ELIMINATION BY ADDITION OR SUBTRACTION.

184. 1. Solve the equations $\begin{cases} 5x-3y=19 & (1) \\ 7x+4y=2 & (2) \end{cases}$

Multiplying (1) by 4, $20x-12y=76$

Multiplying (2) by 3, $21x+12y=6$

Adding these equations, $41x=82$

Whence, $x=2$

Substituting the value of x in (1), $10-3y=19$

$-3y=9$

Whence, $y=-3$

Ans. $x=2,\ y=-3.$

This solution is an example of elimination by *addition*.

2. Solve the equations $\begin{cases} 15x + 8y = 1 & (1) \\ 10x - 7y = -24 & (2) \end{cases}$

$$\begin{array}{lrl} \text{Multiplying (1) by 2,} & 30x + 16y = & 2 \quad (3) \\ \text{Multiplying (2) by 3,} & 30x - 21y = & -72 \quad (4) \\ \hline \textit{Subtracting} \text{ (4) from (3),} & 37y = & 74 \\ & y = & 2 \\ \text{Substituting this value in (2),} & 10x - 14 = & -24 \\ & 10x = & -10 \\ & x = & -1 \end{array}$$

Ans. $x = -1$, $y = 2$.

This solution is an example of elimination by *subtraction.*

RULE.

Multiply the given equations by such numbers as will make the coefficients of one of the unknown quantities equal. Add or subtract the resulting equations according as the equal coefficients have unlike or like signs.

Note. If the coefficients which are to be made equal are prime to each other, each may be used as the multiplier for the other equation. If they are not prime, such multipliers should be used as will produce their lowest common multiple.

Thus, in Ex. 1, to make the coefficients of y equal, we multiply (1) by 4, and (2) by 3. But in Ex. 2, to make the coefficients of x equal, since the L.C.M. of 15 and 10 is 30, we multiply (1) by 2, and (2) by 3.

EXAMPLES.

Solve the following by the method of addition or subtraction:

3. $\begin{cases} 7x + 2y = 31. \\ 3x - 4y = 23. \end{cases}$

4. $\begin{cases} 3x + 7y = 33. \\ x + 2y = 10. \end{cases}$

5. $\begin{cases} 2x - 3y = 4. \\ 6x - y = 28. \end{cases}$

6. $\begin{cases} 7y - 5x = -11. \\ 15x - 14y = 82. \end{cases}$

7. $\begin{cases} 2x - 3y = -24. \\ 3x + 2y = 3. \end{cases}$

8. $\begin{cases} 9x - 13y = 76. \\ 15x + 4y = 101. \end{cases}$

9. $\begin{cases} 24x + 13y = -27. \\ 36x + 11y = -15. \end{cases}$

10. $\begin{cases} 15y - 8x = 12. \\ 25y + 12x = 1. \end{cases}$

11. $\begin{cases} 5x - 7y = 15. \\ 3x - 5y = 13. \end{cases}$

12. $\begin{cases} 7x - 11y = -58. \\ 15x + 8y = 2. \end{cases}$

13. $\begin{cases} 11y - 18x = 2. \\ 24x - 5y = -22. \end{cases}$

14. $\begin{cases} 24x - 18y = -43. \\ 42x + 30y = 17. \end{cases}$

15. $\begin{cases} 11x - 12y = -32. \\ 11y - 12x = 14. \end{cases}$

16. $\begin{cases} 9x - 11y = 24. \\ 10x + 9y = -37. \end{cases}$

17. $\begin{cases} 12x + 21y = -23. \\ 15x + 28y = -30. \end{cases}$

ELIMINATION BY SUBSTITUTION.

185. 1. Solve the equations $\begin{cases} 7x - 3y = -62 & (1) \\ 2y - 5x = 44 & (2) \end{cases}$

Transposing $5x$ in (2), $2y = 5x + 44$

Or, $y = \dfrac{5x + 44}{2}$ (3)

Substituting this in (1), $7x - 3\left(\dfrac{5x+44}{2}\right) = -62$

Or, $7x - \dfrac{15x + 132}{2} = -62$

Clearing of fractions, $14x - 15x - 132 = -124$

$-x = 8$

Whence, $x = -8$

Substituting this value in (3), $y = \dfrac{-40+44}{2} = 2$

Ans. $x = -8$, $y = 2$.

RULE.

Find the value of one of the unknown quantities in terms of the other from one of the given equations, and substitute this value for that quantity in the other equation.

EXAMPLES.

Solve the following by the method of substitution:

2. $\begin{cases} x + y = 7. \\ 3x + 2y = 19. \end{cases}$

3. $\begin{cases} 3x - y = 10. \\ x + 4y = -1. \end{cases}$

4. $\begin{cases} 3x - 4y = 2. \\ 2x - 5y = 6. \end{cases}$

5. $\begin{cases} 7x - 2y = 8. \\ 8y - 5x = -9. \end{cases}$

6. $\begin{cases} 9x - 4y = -4. \\ 15x + 8y = -3. \end{cases}$

7. $\begin{cases} 2x - 7y = 8. \\ 4y - 9x = 19. \end{cases}$

8. $\begin{cases} 5x + 7y = -19. \\ 4x + 5y = -14. \end{cases}$

9. $\begin{cases} 10x - 7y = 9. \\ 4y - 15x = -7. \end{cases}$

10. $\begin{cases} 6x - 5y = -7. \\ 10x + 3y = 11. \end{cases}$

11. $\begin{cases} 9x + 2y = 15. \\ 4x + 7y = 3. \end{cases}$

12. $\begin{cases} 8x + 7y = -23. \\ 5y - 12x = -12. \end{cases}$

13. $\begin{cases} 7y - 3x = 139. \\ 2x + 5y = 91. \end{cases}$

ELIMINATION BY COMPARISON.

186. 1. Solve the equations
$$\begin{cases} 2x - 5y = -16 & (1) \\ 3x + 7y = 5 & (2) \end{cases}$$

Transposing $-5y$ in (1), $2x = 5y - 16$

Or, $$x = \frac{5y - 16}{2} \qquad (3)$$

Transposing $7y$ in (2), $3x = 5 - 7y$

Or, $$x = \frac{5 - 7y}{3}$$

Equating these values of x, $\frac{5y-16}{2}=\frac{5-7y}{3}$

Clearing of fractions, $15y-48=10-14y$

$$29y=58$$

$$y=2$$

Substituting this value in (3), $x=\frac{10-16}{2}=-3$

Ans. $x=-3$, $y=2$.

RULE.

Find the value of the same unknown quantity in terms of the other from each of the given equations, and place these values equal to each other.

EXAMPLES.

Solve the following by the method of comparison:

2. $\begin{cases} x-y=-1. \\ 3x+5y=21. \end{cases}$

3. $\begin{cases} 6x+5y=-8. \\ 4x+3y=-5. \end{cases}$

4. $\begin{cases} 3x-5y=25. \\ 7y-2x=-24. \end{cases}$

5. $\begin{cases} 3x-10y=-36. \\ 2x-9y=-31. \end{cases}$

6. $\begin{cases} 3x-5y=51. \\ 2x+7y=3. \end{cases}$

7. $\begin{cases} 7x+y=-3. \\ x+6y=23. \end{cases}$

8. $\begin{cases} 5x+6y=24. \\ 9y-8x=-26. \end{cases}$

9. $\begin{cases} 7x-8y=-11. \\ x-12y=12. \end{cases}$

10. $\begin{cases} 5x-12y=7. \\ 10x-9y=4. \end{cases}$

11. $\begin{cases} 7y-12x=17. \\ 8x+11y=20. \end{cases}$

12. $\begin{cases} 7x+3y=6. \\ 11x+9y=8. \end{cases}$

13. $\begin{cases} 15x+6y=-7. \\ 8y-21x=18. \end{cases}$

MISCELLANEOUS EXAMPLES.

187. Before applying either method of elimination, each of the given equations should be reduced to its simplest form.

1. Solve the equations

$$\begin{cases} \dfrac{7}{x+3} - \dfrac{3}{y+4} = 0 & (1) \\ x(y-2) - y(x-5) = -13 & (2) \end{cases}$$

From (1),

$$7y + 28 - 3x - 9 = 0, \text{ or } 7y - 3x = -19 \qquad (3)$$

From (2),

$$xy - 2x - xy + 5y = -13, \text{ or } 5y - 2x = -13 \qquad (4)$$

Multiplying (3) by 2, $\quad 14y - 6x = -38 \qquad (5)$

Multiplying (4) by 3, $\quad 15y - 6x = -39 \qquad (6)$

Subtracting (5) from (6), $\quad y = -1$

Substituting in (4), $\quad -5 - 2x = -13$

$$-2x = -8$$

$$x = 4.$$

Ans. $x = 4$, $y = -1$.

Solve the following:

2. $\begin{cases} 11y + 6x = 115. \\ \dfrac{2x}{3} - \dfrac{11y}{6} = -\dfrac{5}{2}. \end{cases}$

3. $\begin{cases} \dfrac{x}{3} + 3y = -46. \\ \dfrac{y}{3} + 3x = 66. \end{cases}$

4. $\begin{cases} \dfrac{3x}{2} - \dfrac{5y}{3} = -4. \\ \dfrac{x}{8} + \dfrac{y}{6} = -4. \end{cases}$

5. $\begin{cases} \dfrac{x}{3} - \dfrac{y}{4} = \dfrac{5}{6}. \\ \dfrac{x}{5} - \dfrac{y}{6} = \dfrac{47}{90}. \end{cases}$

6. $\begin{cases} .2x - .05y = .25. \\ .03x + .3y = .96. \end{cases}$

7. $\begin{cases} .5x + 2y = .01. \\ .11x + .3y = -.009. \end{cases}$

8. $\begin{cases} \dfrac{x+y}{2} - \dfrac{x-y}{3} = 8. \\ \dfrac{x+y}{3} + \dfrac{x-y}{4} = 11. \end{cases}$

9. $\begin{cases} 10 - \dfrac{2x+3y}{5} = \dfrac{y}{3}. \\ \dfrac{4y-3x}{6} = \dfrac{3x}{4} + 1. \end{cases}$

10. $\begin{cases} x(2y-3) = 2y(x+1). \\ \dfrac{3}{x-1} + \dfrac{5}{y+2} = 0. \end{cases}$

11. $\begin{cases} x(y-3) - y(x+4) = 22. \\ (y+1)(x-2) - (y+3)(x-4) = 6. \end{cases}$

12. $\begin{cases} \dfrac{x+y}{x-y} = \dfrac{5}{3}. \\ \dfrac{x+y+1}{x-y-1} = 7. \end{cases}$

15. $\begin{cases} \dfrac{2x+3y}{x+y+13} = -\dfrac{1}{2}. \\ \dfrac{5x}{3} - \dfrac{7y-2}{5} = 11. \end{cases}$

13. $\begin{cases} \dfrac{x}{2} - 12 = \dfrac{y}{4} + 8. \\ \dfrac{x+y}{5} - \dfrac{2y-x}{4} = 15. \end{cases}$

16. $\begin{cases} \dfrac{3x+7}{6} - \dfrac{7-2y}{10} = x. \\ \dfrac{2y-3}{6} - \dfrac{5-3x}{8} = y. \end{cases}$

14. $\begin{cases} x - \dfrac{3x+2}{5} = \dfrac{y+2}{3}. \\ y - \dfrac{2y+1}{3} = \dfrac{x-6}{5}. \end{cases}$

17. $\begin{cases} \dfrac{x+3y}{2x-y} = -\dfrac{3}{8}. \\ \dfrac{7y-x}{2+x+2y} = -17. \end{cases}$

18. $\begin{cases} \dfrac{x-5}{4} - \dfrac{2x-y-1}{3} = \dfrac{2y-2}{5}. \\ \dfrac{2y+x-1}{9} = \dfrac{x+y}{4}. \end{cases}$

19. $$\begin{cases} \dfrac{\dfrac{3x}{4}-\dfrac{y}{3}}{\dfrac{1}{2}}-\dfrac{\dfrac{x}{2}+\dfrac{2y}{5}}{\dfrac{13}{4}}=-\dfrac{7}{6}. \\ 4y-3x=11. \end{cases}$$

20. $$\begin{cases} \dfrac{3x-5y}{2}+3=\dfrac{2x+y}{5}. \\ 8-\dfrac{x-2y}{4}=\dfrac{x}{2}+\dfrac{y}{3}. \end{cases}$$

21. $$\begin{cases} x-\dfrac{2x+y}{3}=\dfrac{17}{12}-\dfrac{2y+x}{4}. \\ \dfrac{5}{4}-\dfrac{2x-y}{4}=y-\dfrac{2y-x}{3}. \end{cases}$$

22. $$\begin{cases} \dfrac{2x}{3}-\dfrac{3y}{5}-\dfrac{x+2y}{4}=3-\dfrac{5x-6y}{4}. \\ \dfrac{x}{2}+y-\dfrac{3x-y}{5}=-5+\dfrac{x}{15}. \end{cases}$$

23. $$\begin{cases} \dfrac{x-2y}{2x-4y-1}=\dfrac{3x}{6x-1}. \\ x-\dfrac{3-5y}{x+2}=\dfrac{4x-13}{4}. \end{cases}$$

24. $$\begin{cases} 4x^2+4xy+272=(x+y)(4x+17). \\ \dfrac{y(x-y)+54}{x-y}=\dfrac{5y+27}{5}. \end{cases}$$

25. $$\begin{cases} x^2-4y^2-17=(x+2y-2)(x-2y+1). \\ \dfrac{xy-5}{y-2}+\dfrac{1-2x}{y-1}=x. \end{cases}$$

Note. In solving literal simultaneous equations, the method of elimination by addition or subtraction is usually to be preferred.

26. Solve the equations $\begin{cases} ax + by = c & (1) \\ a'x + b'y = c' & (2) \end{cases}$

Multiplying (1) by b', $ab'x + bb'y = b'c$

Multiplying (2) by b, $a'bx + bb'y = bc'$

Subtracting, $ab'x - a'bx = b'c - bc'$

Whence, $$x = \frac{b'c - bc'}{ab' - a'b}$$

Multiplying (1) by a', $aa'x + a'by = a'c$ (3)

Multiplying (2) by a, $aa'x + ab'y = ac'$ (4)

Subtracting (3) from (4), $ab'y - a'by = ac' - a'c$

Whence, $$y = \frac{ac' - a'c}{ab' - a'b}$$

Solve the following equations:

27. $\begin{cases} 2x - 3y = a. \\ 3x + 4y = b. \end{cases}$

28. $\begin{cases} ax + by = m. \\ cx + dy = n. \end{cases}$

29. $\begin{cases} ax - by = c. \\ x - y = d. \end{cases}$

30. $\begin{cases} ax - by = 0. \\ mx + ny = p. \end{cases}$

31. $\begin{cases} ax + by = m. \\ cx - dy = n. \end{cases}$

32. $\begin{cases} mx - ny = p. \\ m'x - n'y = p'. \end{cases}$

33. $\begin{cases} \dfrac{x}{a} - \dfrac{y}{b} = m. \\ \dfrac{x}{c} + \dfrac{y}{d} = n. \end{cases}$

34. $\begin{cases} x + ay = a(a + 2b). \\ y - \dfrac{x}{b} = b. \end{cases}$

35. $\begin{cases} ax + by = 2. \\ ab(ay - bx) = a^2 - b^2. \end{cases}$

36. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = 2ab. \\ x + y = ab(a + b). \end{cases}$

37. $\begin{cases} mx + ny = \dfrac{m^4 + n^4}{m^2n^2}. \\ nx + my = \dfrac{m^2 + n^2}{mn}. \end{cases}$

38. $\begin{cases} (a+b)x - (a-b)y = 4ab. \\ (a-b)x - (a+b)y = 0. \end{cases}$

39. $\begin{cases} \dfrac{x+a}{b} + \dfrac{y+b}{a} = \dfrac{2(a^2+b^2)}{ab}. \\ \dfrac{x-b}{a} - \dfrac{y-a}{b} = \dfrac{a^2-b^2}{ab}. \end{cases}$

40. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{m+n} = \dfrac{a^2+m^2-n^2}{a(m+n)}. \\ (m+n)^2(m-n)x = a^3y. \end{cases}$

41. $\begin{cases} \dfrac{x}{a+b} + \dfrac{y}{a-b} = \dfrac{1}{a^2-b^2}. \\ \dfrac{x}{a-b} + \dfrac{y}{a+b} = \dfrac{1}{a^2-b^2}. \end{cases}$

42. $\begin{cases} \dfrac{x}{a+b} + \dfrac{y}{a-b} = 2a. \\ x - y = 4ab. \end{cases}$

Note. Certain fractional equations, in which the unknown quantities occur in the denominators, are readily solved without previously clearing of fractions.

43. Solve the equations $\begin{cases} \dfrac{10}{x} - \dfrac{9}{y} = 8 & (1) \\ \dfrac{8}{x} + \dfrac{15}{y} = -1 & (2) \end{cases}$

Multiplying (1) by 5, $\dfrac{50}{x} - \dfrac{45}{y} = 40$

Multiplying (2) by 3, $\dfrac{24}{x} + \dfrac{45}{y} = -3$

Adding, $\dfrac{74}{x} = 37$

$$37x = 74$$

$$x = 2$$

Substituting in (1),
$$5 - \frac{9}{y} = 8$$

$$-\frac{9}{y} = 3$$

$$y = -3$$

Ans. $x = 2$, $y = -3$.

Solve the following equations:

44. $\begin{cases} \dfrac{3}{x} + \dfrac{1}{y} = \dfrac{5}{4}. \\ \dfrac{2}{x} - \dfrac{3}{y} = -1. \end{cases}$

45. $\begin{cases} \dfrac{2}{x} - \dfrac{3}{y} = -\dfrac{7}{5}. \\ \dfrac{15}{x} - \dfrac{8}{y} = -\dfrac{17}{3}. \end{cases}$

46. $\begin{cases} \dfrac{11}{x} - \dfrac{7}{y} = \dfrac{3}{2}. \\ \dfrac{2}{x} + \dfrac{4}{y} = -5. \end{cases}$

47. $\begin{cases} \dfrac{3}{x} - \dfrac{5}{2y} = 16. \\ \dfrac{1}{2x} + \dfrac{4}{y} = -15. \end{cases}$

48. $\begin{cases} \dfrac{m}{x} + \dfrac{n}{y} = 1. \\ \dfrac{n}{x} + \dfrac{m}{y} = 1. \end{cases}$

49. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = m. \\ \dfrac{c}{x} + \dfrac{d}{y} = n. \end{cases}$

50. $\begin{cases} \dfrac{2}{9x} - \dfrac{5}{2y} = -3. \\ \dfrac{5}{3x} + \dfrac{1}{4y} = \dfrac{17}{6}. \end{cases}$

51. $\begin{cases} \dfrac{m^2}{x} + \dfrac{n^2}{y} = mn(m+n). \\ \dfrac{n}{x} + \dfrac{m}{y} = m^2 + n^2. \end{cases}$

XV. SIMPLE EQUATIONS.

CONTAINING MORE THAN TWO UNKNOWN QUANTITIES.

188. If there are *three* simple equations containing *three* unknown quantities, we may combine two of them by the methods of elimination explained in the last chapter, so as to obtain an equation containing only two unknown quantities. We may then combine the third equation with either of the others, and obtain another equation containing the same two unknown quantities. By solving the equations thus obtained, we derive the values of two of the unknown quantities. These values being substituted in either of the given equations, the value of the third unknown quantity may be determined.

A similar method may be used when the number of equations and of unknown quantities is greater than three.

The method of elimination by addition or subtraction is usually the most convenient.

189. 1. Solve the equations

$$\begin{cases} 6x - 4y - 7z = 17 & (1) \\ 9x - 7y - 16z = 29 & (2) \\ 10x - 5y - 3z = 23 & (3) \end{cases}$$

Multiplying (1) by 3, $18x - 12y - 21z = 51$

Multiplying (2) by 2, $18x - 14y - 32z = 58$

Subtracting, $2y + 11z = -7$ (4)

Multiplying (1) by 5, $30x - 20y - 35z = 85$ (5)

Multiplying (3) by 3, $30x - 15y - 9z = 69$ (6)

Subtracting (5) from (6), $5y + 26z = -16$ (7)

Multiplying (4) by 5, $10y + 55z = -35$

Multiplying (7) by 2, $10y + 52z = -32$

Subtracting, $3z = -3$

$z = -1$

Substituting in (4), $2y - 11 = -7$

$\therefore y = 2$

Substituting the values of y and z in (1),

$$6x - 8 + 7 = 17$$

$$\therefore x = 3$$

Ans. $x = 3,\ y = 2,\ z = -1.$

In certain cases the solution may be abridged by aid of the artifice which is employed in the following example.

2. Solve the equations
$$\begin{cases} u + x + y = 6 & (1) \\ x + y + z = 7 & (2) \\ y + z + u = 8 & (3) \\ z + u + x = 9 & (4) \end{cases}$$

Adding the given equations,

$$3u + 3x + 3y + 3z = 30$$

Whence, $u + x + y + z = 10$ (5)

Subtracting (2) from (5), $u = 3$

Subtracting (3) from (5), $x = 2$

Subtracting (4) from (5), $y = 1$

Subtracting (1) from (5), $z = 4$

EXAMPLES.

Solve the following equations:

3. $\begin{cases} x + y = 2. \\ y + z = -1. \\ z + x = 3. \end{cases}$

4. $\begin{cases} 2x - 5y = -19. \\ 3y + 4z = 13. \\ 2z - 5x = 12. \end{cases}$

5. $\begin{cases} 3x - 2y = -1. \\ 5y + 4z = -6. \\ x - y - 3z = 11. \end{cases}$

6. $\begin{cases} 2x - y = 5. \\ 3x + 2y - z = 6. \\ x - 3y + 2z = 1. \end{cases}$

7. $\begin{cases} x + y + z = 53. \\ x + 2y + 3z = 107. \\ x + 3y + 4z = 137. \end{cases}$

8. $\begin{cases} 3x - y - 2z = -23. \\ 6x + 2y + 3z = 15. \\ 4x + 3y - z = -6. \end{cases}$

9. $\begin{cases} x + y - z = 3. \\ y + z - x = 1. \\ z + x - y = -11. \end{cases}$

10. $\begin{cases} x - 2y + 3z = 0. \\ y - 2z + 3x = -25. \\ z - 2x + 3y = 9. \end{cases}$

11. $\begin{cases} 5x - 3y + 2z = 41. \\ 2x + y - z = 17. \\ 5x + 4y - 2z = 36. \end{cases}$

12. $\begin{cases} 2x + y + z = -2. \\ x + 2y + z = 0. \\ x + y + 2z = -4. \end{cases}$

13. $\begin{cases} 7x + 4y - z = -50. \\ 4x - 5y - 3z = 20. \\ x - 3y - 4z = 30. \end{cases}$

14. $\begin{cases} x - 6y + 4z = 3. \\ 4x + 4y - 3z = 10. \\ 2x + y + 6z = 46. \end{cases}$

15. $\begin{cases} 8x - 9y - 7z = -36. \\ 12x - y - 3z = 36. \\ 6x - 2y - z = 10. \end{cases}$

16. $\begin{cases} 4x - 3y + 2z = 40. \\ 5x + 9y - 7z = 47. \\ 9x + 8y - 3z = 97. \end{cases}$

17. $\begin{cases} \dfrac{x}{2} + \dfrac{y}{3} - \dfrac{z}{4} = -43. \\ \dfrac{x}{3} - \dfrac{y}{4} + \dfrac{z}{2} = 34. \\ \dfrac{x}{4} + \dfrac{y}{2} - \dfrac{z}{3} = -50. \end{cases}$

18. $\begin{cases} 2u - 3x = 1. \\ 3x - 4y = -1. \\ 4y - 5z = 1. \\ 5z - 6u = -2. \end{cases}$

19. $\begin{cases} 2y + z + 2u = -23. \\ y + 3z = -2. \\ 4x + z = 13. \\ \dfrac{x}{3} + 3u = -20. \end{cases}$

20. $\begin{cases} \frac{1}{x}+\frac{1}{y}=1. \\ \frac{1}{y}+\frac{1}{z}=\frac{3}{2}. \\ \frac{1}{z}+\frac{1}{x}=2. \end{cases}$

21. $\begin{cases} \frac{3}{x}-\frac{2}{y}=-13. \\ \frac{3}{y}+\frac{2}{z}=14. \\ \frac{3}{z}-\frac{2}{x}=18. \end{cases}$

22. $\begin{cases} ax+a^2y=2. \\ a^2y+a^3z=2. \\ a^3z+a^4x=a^3+1. \end{cases}$

23. $\begin{cases} 3u-z=22-x-2y. \\ 4x-y=35-3z. \\ 4u-2y=19-3x. \\ z=39-2u-4y. \end{cases}$

24. $\begin{cases} \frac{1}{x}+\frac{2}{y}+\frac{3}{z}=-7. \\ \frac{2}{x}-\frac{3}{y}+\frac{4}{z}=9. \\ \frac{3}{x}+\frac{1}{y}-\frac{2}{z}=5. \end{cases}$

25. $\begin{cases} y-z-\frac{x+z}{2}=1. \\ \frac{x-y}{5}-\frac{x-z}{6}=0. \\ \frac{y+z}{4}-\frac{x+y}{2}=-4. \end{cases}$

26. $\begin{cases} ay+bx=c. \\ cx+az=b. \\ bz+cy=a. \end{cases}$

27. $\begin{cases} 2-z-\frac{3x+y}{4}=0. \\ 8-\frac{y+16z}{3}=3x. \\ 25-12(x+z)=-y. \end{cases}$

28. $\begin{cases} \frac{2x+y}{4}-\frac{y-2z}{3}=1. \\ \frac{x+3y}{3}-\frac{x-z}{4}=-2. \\ \frac{z+y}{3}-\frac{z+x}{4}=-\frac{3}{2}. \end{cases}$

29. * $\begin{cases} ax+y-z=a^2+a-1. \\ ay+z-x=a^2-a+1. \\ az+x-y=a. \end{cases}$

30. $\begin{cases} x-ay+a^2z=a^3. \\ x-by+b^2z=b^3. \\ x-cy+c^2z=c^3. \end{cases}$

* Add the equations together.

XVI. PROBLEMS.

LEADING TO SIMPLE EQUATIONS CONTAINING MORE THAN ONE UNKNOWN QUANTITY.

190. In solving problems where more than one letter is used to represent the unknown quantities, we must obtain from the conditions of the problem *as many independent equations as there are unknown quantities to be determined.*

1. Divide 81 into two parts such that $\frac{3}{5}$ the greater shall exceed $\frac{5}{9}$ the less by 7.

Let $x =$ the greater part,

and $y =$ the less.

By the conditions, $\left\{\begin{array}{l} x + y = 81 \\ \dfrac{3x}{5} = \dfrac{5y}{9} + 7. \end{array}\right.$

Solving these equations, $x = 45$, $y = 36$.

2. If 3 be added to both numerator and denominator of a fraction, its value is $\frac{2}{3}$; and if 2 be subtracted from both numerator and denominator, its value is $\frac{1}{2}$. Required the fraction.

Let $x =$ the numerator,

and $y =$ the denominator.

By the conditions, $\left\{\begin{array}{l} \dfrac{x+3}{y+3} = \dfrac{2}{3} \\ \dfrac{x-2}{y-2} = \dfrac{1}{2}. \end{array}\right.$

Solving these equations, $x = 7$, $y = 12$.

Therefore the fraction is $\dfrac{7}{12}$.

PROBLEMS.

3. Divide 50 into two parts such that three-eighths of the greater shall be equal to two-thirds of the less.

4. Find two numbers such that 7 times the greater exceeds $\frac{1}{7}$ the less by 97, and 7 times the less exceeds $\frac{1}{7}$ the greater by 47.

5. If one-fifth of A's age were added to two-thirds of B's, the sum would be $19\frac{1}{3}$ years; and if two-fifths of B's age were subtracted from seven-eighths of A's, the remainder would be $18\frac{1}{4}$ years. Required their ages.

6. If 1 be added to the numerator of a certain fraction, its value is $\frac{1}{3}$; and if 1 be added to its denominator, its value is $\frac{1}{4}$. Required the fraction.

7. A gentleman at the time of his marriage, found that his wife's age was $\frac{3}{4}$ of his own; but after they had been married 12 years, her age was $\frac{5}{6}$ of his. Required their ages at the time of their marriage.

8. A and B engaged in trade, A with \$240 and B with \$96. A lost twice as much as B; and on settling their accounts, it appeared that A had three times as much remaining as B. How much did each lose?

9. Eight years ago, A was 4 times as old as B; but in 12 years he will be only twice as old. Required their ages at present.

10. If 5 be added to both terms of a fraction, its value is $\frac{1}{2}$; and if 3 be subtracted from both, its value is $\frac{1}{4}$. Required the fraction.

11. A and B agreed to dig a well in 10 days; but having labored together 4 days, B agreed to finish the job, which he did in 16 days. In how many days could each of them alone dig the well?

12. If the greater of two numbers be divided by the less, the quotient is 2 and the remainder 12; but if 4 times the less be divided by the greater, the quotient is 1 and the remainder 14. Required the numbers.

13. If the numerator of a fraction be doubled, and the denominator increased by 7, its value is $\frac{2}{3}$; and if the denominator be doubled, and the numerator increased by 2, the value is $\frac{3}{5}$. Required the fraction.

14. If $a-1$ be subtracted from the numerator of a certain fraction, its value is $a+1$; and if a be added to its denominator, its value is a. Required the fraction.

15. A gentleman's two horses, with their harness, cost $300. The value of the poorer horse, with the harness, was $20 less than the value of the better horse; and the value of the better horse, with the harness, was twice that of the poorer horse. What was the value of each?

16. A merchant has three kinds of sugar. He sells 3 lbs. of the first quality, 4 lbs. of the second, and 2 lbs. of the third, for 60 cents; or, 4 lbs. of the first quality, 1 lb. of the second, and 5 lbs. of the third, for 59 cents; or, 1 lb. of the first quality, 10 lbs. of the second, and 3 lbs. of the third, for 90 cents. Required the price per pound of each quality.

17. A sum of money was divided equally between a certain number of persons. Had there been 3 more, each would have received $1 less; had there been 6 less, each would have received $5 more. How many persons were there, and how much did each receive?

Let $x=$ the number of persons,
and $y=$ what each received.
Then, $xy=$ the sum divided.

By the conditions,

$$\begin{cases}(x+3)(y-1)=xy\\(x-6)(y+5)=xy.\end{cases}$$

Solving these equations, $x=12$, $y=5$.

18. A boy spent his money for oranges. If he had got five more for his money, they would have cost a half-cent each less; if three less, they would have cost a half-cent each more. How much money did he spend, and how many oranges did he get?

19. A merchant has two kinds of grain, worth 60 and 90 cents per bushel respectively. How many bushels of each kind must he take to make a mixture of 40 bushels, worth 80 cents per bushel?

20. My income and assessed taxes together amount to $50. If the income tax were increased 50 per cent, and the assessed tax diminished 25 per cent, they would together amount to $52.50. Required the amount of each tax.

21. A man purchased a certain number of eggs. If he had bought 20 more for the same money, they would have cost a cent apiece less; if 15 less, a cent apiece more. How many eggs did he buy, and at what price?

22. If a certain lot of land were 8 feet longer and 2 feet wider, it would contain 656 square feet more; and if it were 2 feet longer and 8 feet wider, it would contain 776 square feet more. Required its length and width.

23. If B gives A $5, they will have equal amounts; but if A gives B $15, B will have $\frac{7}{9}$ as much as A. How much money has each?

24. Find three numbers such that the first with half the other two, the second with one-third the other two, and the third with one-fourth the other two, may each be equal to 34.

25. There are four numbers whose sum is 136. Twice the first exceeds the second by 46, twice the second exceeds the third by 44, and twice the third exceeds the fourth by 40. Required the numbers.

26. The sum of the digits of a number of three figures is 13. If the number, decreased by 8, be divided by the sum of its second and third digits, the quotient is 25; and if 99 be added to the number, the digits will be inverted. Required the number.

Let $x =$ the first digit,

$y =$ the second,

and $z =$ the third.

Then, $100x + 10y + z =$ the number,

and $100z + 10y + x =$ the number with its digits inverted.

By the conditions,

$$x + y + z = 13,$$

$$\frac{100x + 10y + z - 8}{y + z} = 25,$$

and $100x + 10y + z + 99 = 100z + 10y + x.$

Solving these equations, $x = 2$, $y = 8$, $z = 3$.

Therefore the number is 283.

27. The sum of the digits of a number of two figures is 11; and if 27 be subtracted from the number, the digits will be inverted. Required the number.

28. The sum of the digits of a number of three figures is 11, and the units' figure is twice the figure in the hundreds' place. If 297 be added to the number, the digits will be inverted. Required the number.

29. A and B can perform a piece of work in 6 days, A and C in 8 days, and B and C in 12 days. In how many days can each of them alone perform it?

30. If I were to make my field 5 rods longer and 4 rods wider, its area would be increased by 240 square rods; but if I were to make its length 4 rods less, and its width 5 rods less, its area would be diminished by 210 square rods. Required its length, width, and area.

31. Find three numbers such that the sum of the first and second is c, of the second and third is a, and of the third and first is b.

32. There is a number of three figures, whose digits have equal differences in their order. If the number be divided by half the sum of its digits, the quotient is 41; and if 396 be added to the number, the digits will be inverted. Required the number.

33. A sum of money is divided equally between a certain number of persons. Had there been m more, each would have received a dollars less; if n less, each would have received b dollars more. How many persons were there, and how much did each receive?

34. A gentleman left a sum of money to be divided between his four sons, so that the share of the eldest should be $\frac{1}{2}$ the sum of the shares of the other three, of the second $\frac{1}{3}$ the sum of the other three, and of the third $\frac{1}{4}$ the sum of the other three. It was found that the share of the eldest exceeded that of the youngest by $140. What was the whole sum, and how much did each receive?

35. A grocer bought a certain number of eggs, part at 2 for 5 cents and the rest at 3 for 8 cents, and paid for the whole $1.71. He sold them at 36 cents a dozen, and made 27 cents by the transaction. How many of each kind did he buy?

36. If a number of two figures be divided by the sum of its digits, the quotient is 7; and if the digits be inverted, the quotient of the resulting number, increased by 6, divided by the sum of the digits, is 5. Required the number.

37. If 45 be added to a certain number of two figures, the digits will be inverted; and if the resulting number be divided by the sum of its digits, the quotient is 7 and the remainder 6. Required the number.

38. A and B can do a piece of work in m days, B and C in n days, and C and A in p days. In what time can each alone perform the work?

39. A crew can row 10 miles in 50 minutes down stream, and 12 miles in an hour and a half against the stream. Find the rate in miles per hour of the current, and of the crew in still water.

Let $x =$ the rate of the crew in still water,

and $y =$ the rate of the current.

Then, $x + y =$ the rate rowing down stream,

and $x - y =$ the rate rowing up stream.

Since the distance divided by the rate gives the time, we have, by the conditions,

$$\begin{cases} \dfrac{10}{x+y} = \dfrac{5}{6} \\ \dfrac{12}{x-y} = \dfrac{3}{2}. \end{cases}$$

Solving these equations, $x = 10$, $y = 2$.

40. A crew can row a miles in b hours down stream, and c miles in d hours against the stream. Find the rate in miles per hour of the current, and of the crew in still water.

41. A boatman can row down stream a distance of 20 miles, and back again, in 10 hours; and he finds that he can row 2 miles against the current in the same time that he rows 3 miles with it. Required his time in going and in returning.

42. A number consists of three digits whose sum is 21. The sum of the first digit and twice the second exceeds the third by 8; and if 198 be added to the number, the digits will be inverted. Required the number.

43. A merchant has two casks of wine. He pours from the first cask into the second as much as the second contained at first; he then pours from the second into the first as much as was left in the first; and again from the first into the second as much as was left in the second. There are now 16 gallons in each cask. How many gallons did each contain at first?

44. A number consists of two figures. If the digits be inverted, the sum of the resulting number and the original number is 121; and if the number be divided by the sum of its digits, the quotient is 5 and the remainder 10. Required the number.

45. A man has \$30,000 invested at a certain rate of interest, and owes \$20,000, on which he pays interest at another rate; and the interest which he receives exceeds that which he pays by \$800. Another man has \$35,000 invested at the second rate of interest, and owes \$24,000, on which he pays interest at the first rate; and the interest which he receives exceeds that which he pays by \$310. What are the two rates of interest?

46. A certain sum of money, at simple interest, amounted in 2 years to \$132, and in 5 years to \$150. Required the sum, and the rate of interest.

47. A certain sum of money, at simple interest, amounted in m years to a dollars, and in n years to b dollars. Required the sum, and the rate of interest.

48. A train running from A to B meets with an accident which causes its speed to be reduced to one-third of what it was before, and it is in consequence 5 hours late. If the accident had happened 60 miles nearer B, the train would have been only 1 hour late. What was the rate of the train before the accident?

Let $3x =$ the rate of the train before the accident.
Then, $x =$ its rate after the accident.
Let $y =$ the distance to B from the point of detention.

By the conditions, $$\frac{y}{x} = \frac{y}{3x} + 5$$

$$\frac{y-60}{x} = \frac{y-60}{3x} + 1.$$

Solving these equations, $x = 10$.

Hence the rate of the train before the accident was 30 miles an hour.

49. A man rows down a stream, whose rate is $3\frac{1}{2}$ miles per hour, for a certain distance in 1 hour and 40 minutes. In returning, it takes him 6 hours and 30 minutes to arrive at a point 2 miles short of his starting-place. Find the distance which he rowed down stream, and his rate of pulling.

50. If a certain number be divided by the sum of its two digits, the quotient is 6 and the remainder 1. If the digits be inverted, the quotient of the resulting number increased by 8, divided by the sum of the digits, is 6. Required the number.

51. A train running from A to B meets with an accident which delays it 30 minutes; after which it proceeds at three-fifths its former rate and arrives at B 2 hours and 30 minutes late. If the accident had occurred 30 miles nearer A, the train would have been 3 hours late. What was the rate of the train before the accident?

52. A, B, and C together have $24. A gives to B and C as much as each of them has; B gives to A and C as much as each of them then has; and C gives to A and B as much as each of them then has. They have now equal amounts. How much did each have at first?

53. A and B are building a fence 126 feet long. After 3 hours, A leaves off, and B finishes the work in 14 hours. If 7 hours had occurred before A left off, B would have finished the work in $4\frac{2}{3}$ hours. How many feet does each build in one hour?

54. Divide 115 into three parts such that the first part increased by 30, twice the second part, increased by 2, and 6 times the third part, increased by 4, may all be equal to each other.

55. Four men, A, B, C, and D, play at cards, B having $1 more than C. After A has won half of B's money, B one-third of C's, and C one-fourth of D's, A, B, and C have each $18. How much had each at first?

56. A gives to B and C as much as each of them has; B gives to A and C as much as each of them then has; and C gives to A and B as much as each of them then has. Each has now $48. How much did each have at first?

57. A, B, and C, were engaged to mow a field. The first day, A worked 2 hours, B 3 hours, and C 5 hours, and together they mowed 1 acre; the second day, A worked 4 hours, B 9 hours, and C 6 hours, and all together mowed 2 acres; the third day, A worked 10 hours, B 12 hours, and C 5 hours, and all together mowed 3 acres. In what time could each alone mow an acre?

58. A man invests $3600, partly in $3\frac{1}{2}$ per cent bonds, and partly in 4 per cent bonds. The income from the $3\frac{1}{2}$ per cent bonds exceeds the income from the 4 per cent bonds by $6. How much has he in each kind of bond?

59. A and B run a race of 480 feet. The first heat, A gives B a start of 48 feet, and beats him by 6 seconds; the second heat, A gives B a start of 144 feet, and is beaten by 2 seconds. How many feet can each run in a second?

60. The fore-wheel of a carriage makes 4 revolutions more than the hind-wheel in going 96 feet; but if the circumference of the fore-wheel were $\frac{3}{2}$ as great, and of the hind-wheel $\frac{4}{3}$ as great, the fore-wheel would make only 2 revolutions more than the hind-wheel in going the same distance. Find the circumference of each wheel.

61. A and B together can do a piece of work in $4\frac{4}{5}$ days; but if A had worked one-half as fast, and B twice as fast, they would have finished it in $4\frac{4}{11}$ days. In how many days could each alone perform the work?

62. A and B run a race of 300 yards. The first heat, A gives B a start of 40 yards, and beats him by 2 seconds; the second heat, A gives B a start of 16 seconds, and is beaten by 36 yards. How many yards can each run in a second?

XVII. INVOLUTION.

191. **Involution** is the process of raising a quantity to any required power.

This is effected, as is evident from Art. 13, by taking the quantity as a factor a number of times equal to the exponent of the required power.

192. If the quantity to be involved is positive, all its powers will evidently be positive; but if it is negative, all its *even* powers will be positive, and all its *odd* powers negative. Thus,

$$(-a)^2=(-a)\times(-a) \qquad\qquad =+a^2$$
$$(-a)^3=(-a)\times(-a)\times(-a) \qquad =-a^3$$
$$(-a)^4=(-a)\times(-a)\times(-a)\times(-a)=+a^4;\ \text{etc.}$$

Hence, *the* EVEN *powers of any quantity are positive; and the* ODD *powers of a quantity have the same sign as the quantity itself.*

INVOLUTION OF MONOMIALS.

193. 1. Find the value of $(5a^2c)^4$.

$$(5a^2c)^4=5a^2c\times 5a^2c\times 5a^2c\times 5a^2c=625a^8c^4,\ \textit{Ans.}$$

2. Find the value of $(-3m^4)^3$.

$$(-3m^4)^3=(-3m^4)\times(-3m^4)\times(-3m^4)=-27m^{12},\ \textit{Ans.}$$

From the above examples we derive the following rule:

Raise the numerical coefficient to the required power, and multiply the exponent of each letter by the exponent of the required power.

Give to every even power the positive sign, and to every odd power the sign of the quantity itself.

EXAMPLES.

Write by inspection the values of the following:

3. $(-ab^2c^3)^4$.
4. $(-5a^2b)^3$.
5. $(x^ny)^m$.
6. $(2mn^2x^3)^6$.
7. $(-b^2c^3)^5$.
8. $(a^2b^2c^m)^n$.
9. $(-5m^3n)^4$.
10. $(4a^2b^3c^4)^3$.
11. $(3a^3b^4c)^6$.
12. $(-6x^ry^{7s})^3$.
13. $(4a^mb^{2n})^5$.
14. $(-7x^6y^3z^{15})^3$.

A fraction is raised to any power by *raising both numerator and denominator to the required power.*

For example, $$\left(-\frac{2x^m}{3y^2}\right)^4=\frac{16x^{4m}}{81y^8}.$$

Write by inspection the values of the following:

15. $\left(\frac{ac^2}{b^3}\right)^4$.
16. $\left(\frac{3a^2b^3}{4xy^4}\right)^3$.
17. $\left(-\frac{4ax^2}{5b}\right)^2$.
18. $\left(\frac{2}{3}a^3x^2\right)^6$.
19. $\left(-\frac{7xy^2}{3n}\right)^3$.
20. $\left(-\frac{bcx^n}{4a^2}\right)^5$.

SQUARE OF A POLYNOMIAL.

194. We find by multiplication:

$$\begin{array}{l}
a+b+c \\
a+b+c \\
\hline
a^2+ab+ac \\
\quad\;\, +ab \qquad\quad +b^2+bc \\
\qquad\qquad +ac \qquad\quad +bc+c^2 \\
\hline
a^2+2ab+2ac+b^2+2bc+c^2
\end{array}$$

This result, for convenience of enunciation, may be written as follows:

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc.$$

In a similar manner, we find:

$$(a+b+c+d)^2 = a^2+b^2+c^2+d^2 \\ +2ab+2ac+2ad+2bc+2bd+2cd;$$

and so on.

We have then the following rule for the square of any polynomial:

Write the square of each term, together with twice its product by each of the following terms.

EXAMPLES.

1. Square $2x^2-3x-5$.

The squares of the terms are $4x^4$, $9x^2$, and 25. Twice the first term into each of the following terms gives the results, $-12x^3$ and $-20x^2$; and twice the second term into the following term gives the result, $30x$. Hence,

$$(2x^2-3x-5)^2 = 4x^4+9x^2+25-12x^3-20x^2+30x$$
$$= 4x^4-12x^3-11x^2+30x+25, \text{ Ans.}$$

Square the following expressions:

2. $a-b+c$.

3. $a+b-c$.

4. $2x^2+x+1$.

5. x^2-3x+1.

6. x^2+4x-2.

7. $2x^2-x-3$.

8. $3a^2-5a+4$.

9. $2x^2+5x-7$.

10. $x+2y-3z$.

11. x^3-2x+5.

12. $2x^3+3x^2+1$.

13. $3a^2-2ab-5b^2$.

14. $4m^2+mn^2-3n^4$.

15. $a-b-c+d$.

16. $a-b+c-d$.

17. $1+x+x^2+x^3$.

18. $3x^3-2x^2-x+4$.

19. x^3-4x^2-2x-3.

CUBE OF A BINOMIAL.

195. We find by multiplication:

$$
\begin{array}{rl}
(a+b)^2 = & a^2 + 2ab \;\; + b^2 \\
& \underline{a \;\; + b \qquad\qquad\quad} \\
& a^3 + 2a^2b + \;\; ab^2 \\
& \qquad\quad\;\; \underline{a^2b + 2ab^2 + b^3} \\
(a+b)^3 = & a^3 + 3a^2b + 3ab^2 + b^3
\end{array}
$$

$$
\begin{array}{rl}
(a-b)^2 = & a^2 - 2ab \;\; + b^2 \\
& \underline{a \;\; - b \qquad\qquad\quad} \\
& a^3 - 2a^2b + \;\; ab^2 \\
& \qquad\; \underline{- \;\; a^2b + 2ab^2 - b^3} \\
(a-b)^3 = & a^3 - 3a^2b + 3ab^2 - b^3
\end{array}
$$

That is,

The cube of the sum of two quantities is equal to the cube of the first, plus three times the square of the first times the second, plus three times the first times the square of the second, plus the cube of the second.

The cube of the difference of two quantities is equal to the cube of the first, minus three times the square of the first times the second, plus three times the first times the square of the second, minus the cube of the second.

EXAMPLES.

1. Find the cube of $a + 2b$.

$$
\begin{aligned}
(a+2b)^3 &= a^3 + 3a^2(2b) + 3a(2b)^2 + (2b)^3 \\
&= a^3 + 6a^2b + 12ab^2 + 8b^3, \text{ } \textit{Ans.}
\end{aligned}
$$

2. Find the cube of $2x - 3y^2$.

$$
\begin{aligned}
(2x-3y^2)^3 &= (2x)^3 - 3(2x)^2(3y^2) + 3(2x)(3y^2)^2 - (3y^2)^3 \\
&= 8x^3 - 36x^2y^2 + 54xy^4 - 27y^6, \text{ } \textit{Ans.}
\end{aligned}
$$

Find the cubes of the following:

3. $x+3$.
4. $2x-1$.
5. $ab-cd$.
6. $a+4b$.
7. $3m^2-1$.
8. x^2+4.
9. $a+5b$.
10. $2x-5y$.
11. $2x^3-3x$.
12. $6x^2+xy$.
13. $3m+5n$.
14. $3xy-4a^2$.

The cube of a trinomial may be found by the above method, if two of its terms be enclosed in a parenthesis and regarded as a single term.

15. Find the cube of x^2-2x-1.

$$\begin{aligned}(x^2-2x-1)^3 &= [(x^2-2x)-1]^3\\ &= (x^2-2x)^3-3(x^2-2x)^2+3(x^2-2x)-1\\ &= x^6-6x^5+12x^4-8x^3-3(x^4-4x^3+4x^2)\\ &\qquad +3(x^2-2x)-1\\ &= x^6-6x^5+9x^4+4x^3-9x^2-6x-1,\ \textit{Ans.}\end{aligned}$$

Find the cubes of the following:

16. x^2-x-1.
17. $a-b+1$.
18. $a+b-c$.
19. x^2-2x+2.
20. x^2+3x+1.
21. $2x^2-3x-1$.

ANY POWER OF A BINOMIAL.

196. By actual multiplication, we obtain:

$$(a+b)^2=a^2+2ab+b^2$$

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3$$

$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4;\ \text{etc.}$$

$$(a-b)^2=a^2-2ab+b^2$$

$$(a-b)^3=a^3-3a^2b+3ab^2-b^3$$

$$(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4;\ \text{etc.}$$

In these results we observe the following laws:

I. The number of terms is one more than the exponent of the binomial.

II. The exponent of a in the first term is the same as the exponent of the binomial, and decreases by 1 in each succeeding term.

III. The exponent of b in the second term is 1, and increases by 1 in each succeeding term.

IV. The coefficient of the first term is 1; and of the second term, is the exponent of the binomial.

V. If the coefficient of any term be multiplied by the exponent of a in that term, and the result divided by the exponent of b increased by 1, the quotient will be the coefficient of the next term.

VI. If the second term of the binomial is negative, the terms in the result are alternately positive and negative.

By aid of the above laws, any power of a binomial may be written by inspection.

EXAMPLES.

1. Expand $(a+x)^5$.

The exponent of a in the first term is 5, and decreases by 1 in each succeeding term.

The exponent of x in the second term is 1, and increases by 1 in each succeeding term.

The coefficient of the first term is 1; of the second term, 5; multiplying the coefficient of the second term by 4, the exponent of a in that term, and dividing the result by the exponent of x increased by 1, or 2, we have 10 for the coefficient of the third term; and so on. Hence,

$$(a+x)^5 = a^5 + 5\,a^4x + 10\,a^3x^2 + 10\,a^2x^3 + 5\,ax^4 + x^5, \ \textit{Ans.}$$

Note. The coefficients of terms equally distant from the beginning and end of the expansion are equal. Thus the coefficients of the latter half of an expansion may be written out from the first half.

2. Expand $(1-x)^6$.

$$\begin{aligned}(1-x)^6 &= 1^6 - 6\cdot 1^5\cdot x + 15\cdot 1^4\cdot x^2 - 20\cdot 1^3\cdot x^3 \\ &\quad + 15\cdot 1^2\cdot x^4 - 6\cdot 1\cdot x^5 + x^6 \\ &= 1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6, \textit{ Ans.}\end{aligned}$$

Note. If the first term of the binomial is numerical, it is convenient to write the exponents at first without reduction. The result should afterwards be reduced to its simplest form.

Expand the following:

3. $(a-b)^5$.

4. $(a+b)^6$.

5. $(a-b)^7$.

6. $(x-1)^5$.

7. $(1-x)^4$.

8. $(x+y)^7$.

9. $(m-n)^6$.

10. $(2+x)^4$.

11. $(x-4)^4$.

12. $(a-3)^5$.

13. $(a+2)^5$.

14. $(x-2)^6$.

15. Expand $(3m-n^2)^4$.

$$\begin{aligned}(3m-n^2)^4 &= [(3m)-(n^2)]^4 \\ &= (3m)^4 - 4(3m)^3(n^2) + 6(3m)^2(n^2)^2 \\ &\quad - 4(3m)(n^2)^3 + (n^2)^4 \\ &= 81m^4 - 108m^3n^2 + 54m^2n^4 - 12mn^6 + n^8, \textit{ Ans.}\end{aligned}$$

Note. If either term of the binomial has a coefficient or exponent other than unity, it should be enclosed in a parenthesis before applying the laws.

Expand the following:

16. $(a-3x)^5$.

17. $(3+2b)^4$.

18. $(a^2+bc)^7$.

19. $(x^3-4)^4$.

20. $(2a^2+b)^6$.

21. $(2m^3-3n^2)^4$.

XVIII. EVOLUTION.

197. If a quantity be resolved into any number of equal factors, one of these factors is called a **Root** of the quantity.

198. **Evolution** is the process of finding any required root of a quantity. This is effected, as is evident from the preceding article, by finding a quantity which, when raised to the proposed power, will produce the given quantity.

199. The **Radical Sign**, $\sqrt{\ }$, when prefixed to a quantity, indicates that some root of the quantity is to be found.

Thus, $\sqrt{a}$ indicates the *second* or *square* root of a;

$\sqrt[3]{a}$ indicates the *third* or *cube* root of a;

$\sqrt[4]{a}$ indicates the *fourth* root of a; and so on.

The *index* of the root is the figure written over the radical sign. When no index is written, the square root is understood.

EVOLUTION OF MONOMIALS.

200. Required the cube root of $a^3b^6c^9$.

By Art. 198, we are to find a quantity which, when raised to the third power, will produce $a^3b^6c^9$. That quantity is evidently ab^2c^3. Hence,

$$\sqrt[3]{a^3b^6c^9} = ab^2c^3.$$

That is, *any root of a monomial is obtained by dividing the exponent of each factor by the index of the required root.*

201. From the relation of a root to its corresponding power, it follows from Art. 192 that:

1. *The odd roots of a quantity have the same sign as the quantity itself.*

Thus, $\sqrt[3]{a^3} = a$, and $\sqrt[5]{-a^5} = -a$.

2. *The even roots of a positive quantity are either positive or negative.*

For the even powers of either a positive or a negative quantity are positive.

Thus, $\sqrt[4]{a^4} = a$ or $-a$; that is, $\sqrt[4]{a^4} = \pm a$.

Note. The sign $\pm$, called the *double sign*, is prefixed to a quantity when we wish to indicate that it is either + or −.

3. *The even roots of a negative quantity are impossible.*

For no quantity when raised to an even power can produce a negative result. Such roots are called *imaginary* quantities.

202. From Arts. 200 and 201 we derive the following rule:

Extract the required root of the numerical coefficient, and divide the exponent of each letter by the index of the root.

Give to every even root of a positive quantity the sign $\pm$, and to every odd root of any quantity the sign of the quantity itself.

Note. Any root of a fraction may be found by taking the required root of each of its terms.

EXAMPLES.

1. Find the square root of $9a^4b^2c^6$.

By the rule, $\sqrt{9a^4b^2c^6} = \pm 3a^2bc^3$, *Ans.*

2. Find the fifth root of $-32a^{10}x^{5m}$.

$$\sqrt[5]{-32a^{10}x^{5m}} = -2a^2x^m, \textit{ Ans.}$$

Find the values of the following:

3. $\sqrt[3]{-125x^3y^6}$.	7. $\sqrt[3]{-8a^3b^6x^9}$.	11. $\sqrt[4]{81m^{16}n^{20}}$.
4. $\sqrt{49a^4b^2c^{12}}$.	8. $\sqrt{121a^{12}c^2}$.	12. $\sqrt[5]{-243c^{5n}d^{10m}}$.
5. $\sqrt[5]{m^{15}n^5p^{10}}$.	9. $\sqrt[m]{a^{mn}b^{mp}}$.	13. $\sqrt[6]{64a^{18}b^{24}c^6}$.
6. $\sqrt[4]{16a^4b^8}$.	10. $\sqrt{81a^{2n}x^{2m+2}}$.	14. $\sqrt[3]{x^{3n+3}y^{9m-6}}$.

15. $\sqrt{\frac{9x^2y^4}{16m^6}}$. 17. $\sqrt[5]{-\frac{32x^{15}}{y^{10}}}$. 19. $\sqrt[3]{-\frac{64m^3n^6}{125}}$.

16. $\sqrt[3]{\frac{8a^3b^9}{27c^6}}$. 18. $\sqrt[4]{\frac{a^4}{81b^8c^4}}$. 20. $\sqrt[5]{\frac{a^{5m}}{243x^{10}}}$.

SQUARE ROOT OF POLYNOMIALS.

203. Since $(a+b)^2 = a^2 + 2ab + b^2$, we know that the square root of $a^2 + 2ab + b^2$ is $a + b$.

It is required to find a process by which, when the quantity $a^2 + 2ab + b^2$ is given, its square root, $a + b$, may be determined.

$$\begin{array}{l|l|l} & a^2+2ab+b^2 & a+b \\ & a^2 & \\ \hline 2a+b & 2ab+b^2 & \\ & 2ab+b^2 & \\ \hline \end{array}$$

The square root of the first term is a, which is the first term of the root. Subtracting its square from the given expression, the remainder is $2ab + b^2$, or $(2a+b)b$. Dividing the first term of this remainder by $2a$, or twice the first term of the root, we obtain b, the second term. This being added to $2a$, gives the complete divisor $2a + b$; which, when multiplied by b, and the product, $2ab + b^2$, subtracted from the remainder, completes the operation.

From the above process we derive the following rule:

Arrange the terms according to the powers of some letter.

Find the square root of the first term, write it as the first term of the root, and subtract its square from the given expression.

Divide the first term of the remainder by twice the first term of the root, and add the quotient to the root and also to the divisor.

Multiply the complete divisor by the term of the root last obtained, and subtract the product from the remainder.

If other terms remain, proceed as before, doubling the part of the root already found for the next trial-divisor.

EXAMPLES.

204. **1.** Find the square root of $9x^4 - 30a^3x^2 + 25a^6$.

$$
\begin{array}{r|l|l}
 & 9x^4 - 30a^3x^2 + 25a^6 & 3x^2 - 5a^3, \ \textit{Ans.} \\
 & 9x^4 & \\
\hline
6x^2 - 5a^3 & -30a^3x^2 + 25a^6 & \\
 & -30a^3x^2 + 25a^6 & \\
\hline
\end{array}
$$

The square root of the first term is $3x^2$, which is the first term of the root. Subtracting $9x^4$ from the given expression, we have $-30a^3x^2$ as the first term of the remainder. Dividing this by twice the first term of the root, $6x^2$, we obtain the second term of the root, $-5a^3$, which, added to $6x^2$, completes the divisor $6x^2 - 5a^3$. Multiplying this divisor by $-5a^3$, and subtracting the product from the remainder, there is no remainder. Hence, $3x^2 - 5a^3$ is the required square root.

2. Find the square root of

$$12x^5 - 14x^3 + 1 - 8x^4 + 9x^6 + 4x.$$

Arranging according to the descending powers of x,

$$
\begin{array}{r|l|l}
 & 9x^6 + 12x^5 - 8x^4 - 14x^3 + 4x + 1 & 3x^3 + 2x^2 - 2x - 1, \\
 & 9x^6 & \textit{Ans.} \\
\hline
6x^3 + 2x^2 & 12x^5 & \\
 & 12x^5 + 4x^4 & \\
\hline
6x^3 + 4x^2 - 2x & -12x^4 & \\
 & -12x^4 - 8x^3 + 4x^2 & \\
\hline
6x^3 + 4x^2 - 4x - 1 & -6x^3 - 4x^2 + 4x + 1 & \\
 & -6x^3 - 4x^2 + 4x + 1 & \\
\hline
\end{array}
$$

It will be observed that each trial-divisor is equal to the preceding complete divisor, with its last term doubled.

Note. Since every square root has the double sign (Art. 201), the result may be written in a different form by changing the sign of each term. Thus, in Example 2, another form of the answer is

$$-3x^3 - 2x^2 + 2x + 1.$$

Find the square roots of the following:

3. $a^4 - 4a^3 + 6a^2 - 4a + 1$.

4. $4x^4 - 4x^3 - 3x^2 + 2x + 1$.

5. $9 - 12x + 10x^2 - 4x^3 + x^4$.

6. $19x^2 + 6x^3 + 25 + x^4 + 30x$.

7. $40x + 25 - 14x^2 + 9x^4 - 24x^3$.

8. $m^2 + 2m - 1 - \frac{2}{m} + \frac{1}{m^2}$.

9. $4a^4 + 64b^4 - 20a^3b - 80ab^3 + 57a^2b^2$.

10. $28x^3 + 4x^4 - 14x + 1 + 45x^2$.

11. $a^2 + b^2 + c^2 - 2ab - 2ac + 2bc$.

12. $x^2 + 4y^2 + 9z^2 - 4xy + 6xz - 12yz$.

13. $9x^6 + 30x^5 + 25x^4 - 42x^3 - 70x^2 + 49$.

14. $16c^6 - 40c^4 - 24c^3 + 25c^2 + 30c + 9$.

15. $9 + a^6 + 30a - 4a^5 + 13a^2 + 14a^4 - 14a^3$.

16. $4x^6 - 4x^5y - 3x^4y^2 - 6x^3y^3 + 5x^2y^4 + 4xy^5 + 4y^6$.

17. $25x^4 - 44x^3 - 40x + 4x^6 + 25 + 46x^2 - 12x^5$.

18. $\frac{a^4}{9} - \frac{2a^3b}{3} + \frac{4a^2b^2}{3} - ab^3 + \frac{b^4}{4}$.

19. $9x^6 - 12x^5y + 10x^4y^2 - 16x^3y^3 + 9x^2y^4 - 4xy^5 + 4y^6$.

Find to four terms the approximate square roots of the following:

20. $1 + x$.

21. $1 - 2a$.

22. $a^2 - 4ab + b^2$.

23. $4x^2 + 2y$.

SQUARE ROOT OF NUMBERS.

205. The method of Art. 204 may be used to extract the square roots of arithmetical numbers.

The square root of 100 is 10; of 10,000 is 100; etc. Hence, the square root of a number less than 100 is less than 10; the square root of a number between 10,000 and 100 is between 100 and 10; and so on.

That is, the integral part of the square root of a number of one or two figures, contains *one* figure; of a number of three or four figures, contains *two* figures; and so on. Hence,

If a point be placed over every second figure in any integral number, beginning with the units' place, the number of points shows the number of figures in the integral part of its square root.

206. Let it be required to find the square root of 4624.

$$
\begin{array}{r|r|l}
 & 4\dot{6}2\dot{4} & 60+8 \\
 & a^2=3600 & =a+b \\
\hline
120+8 & 1024 & \\
=2a+b & 1024 & \\
\cline{2-2}
\end{array}
$$

Pointing the number according to the rule of Art. 205, we see that there are two figures in the integral part of the square root.

Let a denote the value of the number in the tens' place in the root, and b the number in the units' place. Then a must be the greatest multiple of 10 whose square is less than 4624; this we find to be 60. Subtracting a^2, that is, the square of 60 or 3600, from the given number, the remainder is 1024. Dividing the remainder by $2a$ or 120, we have 8 as the value of b. Adding this to 120, multiplying the result by 8, and subtracting the product, 1024, there is no remainder. Hence, $60+8$ or 68 is the required square root.

The ciphers being omitted for the sake of brevity, the work will stand as follows:

$$
\begin{array}{r|r|l}
 & 4\dot{6}2\dot{4} & 68 \\
 & 36 & \\
\hline
128 & 1024 & \\
 & 1024 & \\
\cline{2-2}
\end{array}
$$

From the above process we derive the following rule:

Separate the number into periods by pointing every second figure, beginning with the units' place.

Find the greatest square in the left-hand period, and write its square root as the first figure of the root; subtract its square from the number, and to the result bring down the next period.

Divide this remainder, omitting the last figure, by twice the part of the root already found, and annex the quotient to the root and also to the divisor.

Multiply the complete divisor by the figure of the root last obtained, and subtract the product from the remainder.

If other periods remain, proceed as before, doubling the part of the root already found for the next trial-divisor.

Note 1. It should be observed that decimals require to be pointed to the *right*.

Note 2. As the trial-divisor is an *incomplete* divisor, it is sometimes found that after completion it gives a product greater than the remainder. In such a case, the last root-figure is too large, and one less must be substituted for it.

Note 3. If any root-figure is 0, annex 0 to the trial-divisor, and bring down to the remainder the next period.

EXAMPLES.

207. 1. Find the square root of 49.449024.

$$\begin{array}{r|l|l} & 4\dot{9}.4\dot{4}9\dot{0}2\dot{4} & 7.032,\ \textit{Ans.} \\ & 49 & \\ \hline 1403 & 4490 & \\ & 4209 & \\ \hline 14062 & 28124 & \\ & 28124 & \\ \hline \end{array}$$

Since the second root-figure is 0, we annex 0 to the trial-divisor 14, and bring down to the remainder the next period, 90.

Extract the square roots of the following:

2. 45796.	**6.** .247009.	**10.** 446.0544.
3. 273529.	**7.** .081796.	**11.** .0022448644.
4. 654481.	**8.** .521284.	**12.** 811440.64.
5. 33.1776.	**9.** 1.170724.	**13.** .68112009.

If there is a final remainder, the given number has no exact square root; but we may continue the operation by annexing periods of ciphers, and thus obtain an approximate value of the square root, correct to any desired number of decimal places.

14. Extract the square root of 12 to five figures.

$$
\begin{array}{r|l}
 & 1\dot{2}.0\dot{0}0\dot{0}0\dot{0}0\dot{0} \;\Big|\; 3.4641\ldots,\ \mathit{Ans.} \\
 & 9 \\
\hline
64 & 300 \\
 & 256 \\
\hline
686 & 4400 \\
 & 4116 \\
\hline
6924 & 28400 \\
 & 27696 \\
\hline
69281 & 70400 \\
 & 69281 \\
\hline
 & 1119
\end{array}
$$

Extract the square roots of the following to five figures:

15. 2.	**18.** 11.	**21.** .7.	**24.** .001.
16. 3.	**19.** 31.	**22.** .08.	**25.** .00625.
17. 5.	**20.** 17.3.	**23.** .144.	**26.** 2.08627.

The square root of a fraction may be obtained by taking the square roots of its terms.

If the denominator is not a perfect square, it is better to reduce the fraction to an equivalent fraction whose denominator is a perfect square.

Thus, to obtain the square root of $\frac{3}{8}$, we should proceed as follows:

$$\sqrt{\frac{3}{8}}=\sqrt{\frac{6}{16}}=\frac{\sqrt{6}}{4}=\frac{2.44949\ldots}{4}=.61237\ldots$$

Extract the square roots of the following to five figures:

27. $\frac{7}{4}$. 29. $\frac{10}{9}$. 31. $\frac{4}{3}$. 33. $\frac{11}{8}$. 35. $\frac{7}{18}$.

28. $\frac{3}{16}$. 30. $\frac{1}{5}$. 32. $\frac{5}{12}$. 34. $\frac{9}{10}$. 36. $\frac{13}{72}$.

CUBE ROOT OF POLYNOMIALS.

208. Since $(a+b)^3=a^3+3a^2b+3ab^2+b^3$, we know that the cube root of $a^3+3a^2b+3ab^2+b^3$ is $a+b$.

It is required to find a process by which, when the quantity $a^3+3a^2b+3ab^2+b^3$ is given, its cube root, $a+b$, may be determined.

$$\begin{array}{r|l|l}
 & a^3+3a^2b+3ab^2+b^3 & a+b \\
 & a^3 & \\
\hline
3a^2+3ab+b^2 & 3a^2b+3ab^2+b^3 & \\
 & 3a^2b+3ab^2+b^3 & \\
\hline
\end{array}$$

The cube root of the first term is a, which is the first term of the root. Subtracting its cube from the given expression, the remainder is $3a^2b+3ab^2+b^3$, or $(3a^2+3ab+b^2)b$. Dividing the first term of this remainder by $3a^2$, or three times the square of the first term of the root, we obtain b, the second term.

Adding to the trial-divisor $3ab$, that is, three times the product of the first term of the root by the second, and b^2, that is, the square of the last term of the root, completes the divisor, $3a^2+3ab+b^2$. This being multiplied by b, and the product, $3a^2b+3ab^2+b^3$, subtracted from the remainder, completes the operation.

From the above process we derive the following rule:

Arrange the terms according to the powers of some letter.

Find the cube root of the first term, write it as the first term of the root, and subtract its cube from the given expression.

Divide the first term of the remainder by three times the square of the first term of the root, and write the quotient as the next term of the root.

Add to the trial-divisor three times the product of the first term of the root by the second, and the square of the second term.

Multiply the complete divisor by the term of the root last obtained, and subtract the product from the remainder.

If other terms remain, proceed as before, taking three times the square of the root already found for the next trial-divisor.

EXAMPLES.

209. 1. Find the cube root of $8x^6 - 36x^4y + 54x^2y^2 - 27y^3$.

$$\begin{array}{r|l|l}
 & 8x^6 - 36x^4y + 54x^2y^2 - 27y^3 & 2x^2 - 3y, \textit{ Ans.} \\
 & 8x^6 & \\
\hline
12x^4 - 18x^2y + 9y^2 & -36x^4y + 54x^2y^2 - 27y^3 & \\
 & -36x^4y + 54x^2y^2 - 27y^3 & \\
\cline{2-2}
\end{array}$$

The cube root of the first term is $2x^2$, which is the first term of the root. Subtracting $8x^6$ from the given expression, we have $-36x^4y$ as the first term of the remainder. Dividing this by three times the square of the first term of the root, $12x^4$, we obtain $-3y$ as the second term of the root. Adding to the trial-divisor three times the product of the first term of the root by the second, $-18x^2y$, and the square of the second term, $9y^2$, completes the divisor, $12x^4 - 18x^2y + 9y^2$. Multiplying this by $-3y$, and subtracting the product from the remainder, there is no remainder. Hence, $2x^2 - 3y$ is the required cube root.

2. Find the cube root of $40x^3 - 6x^5 - 96x + x^6 - 64$.

Arranging according to the descending powers of x,

$$
\begin{array}{l|l|l}
 & x^6-6x^5+40x^3-96x-64 & x^2-2x-4, \\
 & x^6 & \qquad\qquad Ans. \\
\hline
3x^4-6x^3+4x^2 & -6x^5 & \\
 & -6x^5+12x^4-8x^3 & \\
\hline
3x^4-12x^3+12x^2 & -12x^4+48x^3-96x-64 & \\
\qquad -12x^2+24x+16 & & \\
\hline
3x^4-12x^3 \qquad +24x+16 & -12x^4+48x^3-96x-64 & \\
\hline
\end{array}
$$

The second complete divisor is formed as follows:

The trial-divisor is 3 times the square of the root already found; that is, $3(x^2-2x)^2$, or $3x^4-12x^3+12x^2$. Three times the product of the root already found by the last term of the root is $3(-4)(x^2-2x)$, or $-12x^2+24x$; and the square of the last root-term is 16. Adding these, we have for the complete divisor $3x^4-12x^3+24x+16$.

Find the cube roots of the following:

3. $1-6y+12y^2-8y^3$.

4. $27x^6+27x^4+9x^2+1$.

5. $54xy^2+27y^3+36x^2y+8x^3$.

6. $64a^3-144a^2xy+108ax^2y^2-27x^3y^3$.

7. $x^6+6x^5-40x^3+96x-64$.

8. $y^6-1+5y^3-3y^5-3y$.

9. $15x^4-6x-6x^5+15x^2+1+x^6-20x^3$.

10. $9x^3-21x^2-36x^5+8x^6-9x+42x^4-1$.

11. $8a^6-12a^5-54a^4+59a^3+135a^2-75a-125$.

12. $30x^2-12x^5-12x+8-25x^3+8x^6+30x^4$.

13. $x^6+3x^5y-3x^4y^2-11x^3y^3+6x^2y^4+12xy^5-8y^6$.

14. $27a^6-54a^5b+9a^4b^2+28a^3b^3-3a^2b^4-6ab^5-b^6$.

CUBE ROOT OF NUMBERS.

210. The method of Art. 209 may be used to extract the cube roots of arithmetical numbers.

The cube root of 1000 is 10; of 1,000,000 is 100; etc. Hence, the cube root of a number less than 1000 is less than 10; the cube root of a number between 1,000,000 and 1000 is between 100 and 10; and so on.

That is, the integral part of the cube root of a number of one, two, or three figures, contains *one* figure; of a number of four, five, or six figures, contains *two* figures; and so on. Hence,

If a point be placed over every third figure in any integral number, beginning with the units' place, the number of points shows the number of figures in the integral part of its cube root.

211. Let it be required to find the cube root of 157464.

$$\begin{array}{rr|l}
 & 15\dot{7}46\dot{4} & 50+4 \\
 & a^3 = 125000 & =a+b \\
\hline
3a^2 = 7500 & 32464 & \\
3ab = 600 & & \\
b^2 = 16 & & \\
\cline{1-1}
8116 & 32464 & \\
\end{array}$$

Pointing the number according to the rule of Art. 210, we see that there are two figures in the integral part of the cube root.

Let a denote the value of the number in the tens' place in the root, and b the number in the units' place. Then a must be the greatest multiple of 10 whose cube is less than 157464; this we find to be 50. Subtracting a^3, that is, the cube of 50 or 125000, from the given number, the remainder is 32464. Dividing this remainder by $3a^2$, that is, 3 times the square of 50 or 7500, we obtain 4 as the value of b. Adding to the trial-divisor $3ab$, that is, 3 times the product of 50 and 4, or 600, and b^2, or 16, we have the complete divisor 8116. Multiplying this by 4, and subtracting the product, 32464, there is no remainder. Hence, $50 + 4$ or 54 is the required cube root.

The ciphers being omitted for the sake of brevity, the work will stand as follows:

$$\begin{array}{r|l|l}
 & 15\dot{7}46\dot{4} & 54 \\
 & 125 & \\
\hline
7500 & 32464 & \\
600 & & \\
16 & & \\
\hline
8116 & 32464 & \\
\hline
\end{array}$$

From the above process, we derive the following rule:

Separate the number into periods by pointing every third figure, beginning with the units' place.

Find the greatest cube in the left-hand period, and write its cube root as the first figure of the root; subtract its cube from the number, and to the result bring down the next period.

Divide this remainder by three times the square of the root already found, with two ciphers annexed, and write the quotient as the next figure of the root.

Add to the trial-divisor three times the product of the last root-figure and the part of the root previously found, with one cipher annexed, and the square of the last root-figure.

Multiply the complete divisor by the figure of the root last obtained, and subtract the product from the remainder.

If other periods remain, proceed as before, taking three times the square of the root already found for the next trial-divisor.

The notes to Art. 206 apply with equal force to examples in cube root, except that in Note 3 *two* ciphers should be annexed to the trial-divisor.

212. In the illustration of Art. 208, if there had been more terms in the given quantity, the next trial-divisor would have been three times the square of $a + b$; that is, $3a^2 + 6ab + 3b^2$. We observe that this is obtained from the preceding complete divisor, $3a^2 + 3ab + b^2$, by adding to it its second term, $3ab$, and twice its third term, $2b^2$. We may

then use the following rule for forming the successive trial-divisors in the cube root of numbers:

To the preceding complete divisor, add its second term and twice its third term; and annex two ciphers to the result.

EXAMPLES.

213. **1.** Find the cube root of 8.144865728.

	8.144865728	2.012, *Ans.*
	8	
120000	144865	
600		
1		
120601	120601	
600	24264728	
2		
12120300		
12060		
4		
12132364	24264728	

Since the second root-figure is 0, we annex two ciphers to the trial-divisor 1200, and bring down to the remainder the next period, 865.

The second trial-divisor is formed by the rule of Art. 212. The preceding complete divisor is 120601; adding its second term, 600, and twice its third term, 2, we have 121203; annexing two ciphers to this, we obtain the result 12120300.

Extract the cube roots of the following:

2. 29791.

3. 97.336.

4. .681472.

5. 1860867.

6. 1.481544.

7. .000941192.

8. 8.242408.

9. 51478848.

10. 10077.696.

11. .517781627.

12. 116.930169.

13. .031855013.

14. .724150792.

15. 1039509.197.

16. .000152273304.

Extract the cube roots of the following to four figures:

17. 2. **19.** 7.2. **21.** $\frac{3}{8}$. **23.** $\frac{7}{27}$.

18. 6. **20.** .03. **22.** $\frac{5}{4}$. **24.** $\frac{2}{3}$.

214. When the index of the required root is the product of two or more numbers, we may obtain the result by *successive extractions of the simpler roots.*

For, by Art. 198, $(\sqrt[mn]{a})^{mn} = a.$

Taking the nth root of both members,

$$(\sqrt[mn]{a})^m = \sqrt[n]{a}. \tag{1}$$

Taking the mth root of both members of (1),

$$\sqrt[mn]{a} = \sqrt[m]{(\sqrt[n]{a})}.$$

That is,

The mnth root of a quantity is equal to the mth root of the nth root of that quantity.

For example, the fourth root is the square root of the square root; the sixth root is the cube root of the square root; etc.

EXAMPLES.

Find the fourth roots of the following:

1. $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4.$

2. $x^8 - 4x^7 + 10x^6 - 16x^5 + 19x^4 - 16x^3 + 10x^2 - 4x + 1.$

3. $x^8 - 8x^7 + 16x^6 + 16x^5 - 56x^4 - 32x^3 + 64x^2 + 64x + 16.$

Find the sixth roots of the following:

4. $a^{12} - 6a^{10} + 15a^8 - 20a^6 + 15a^4 - 6a^2 + 1.$

5. $64x^6 + 192x^5 + 240x^4 + 160x^3 + 60x^2 + 12x + 1.$

XIX. THE THEORY OF EXPONENTS.

215. In the preceding chapters we have considered an exponent only as a positive whole number. It is, however, found convenient to employ fractional and negative exponents; and we proceed to define them, and to prove the rules for their use.

216. In Art. 13 we defined a positive integral exponent as indicating how many times a quantity was taken as a factor; thus,

$$a^m \text{ signifies } a \times a \times a \times \cdots \text{ to } m \text{ factors.}$$

We have also found the following rules to hold when m and n are positive integers:

$$\text{I. } a^m \times a^n = a^{m+n}. \qquad \text{(Art. 79.)}$$

$$\text{II. } (a^m)^n = a^{mn}. \qquad \text{(Art. 193.)}$$

217. The definition of Art. 13 has no meaning unless the exponent is a positive integer, and we must therefore adopt new definitions for fractional and negative exponents. It is convenient to have all forms of exponents subject to the same laws in regard to multiplication, division, etc., and we shall therefore assume Rule I. to hold for *all* values of m and n, and find what meanings must be attached to fractional and negative exponents in consequence.

218. Required the meaning of $a^{\frac{5}{3}}$.

Since Rule I. is to hold universally, we must have

$$a^{\frac{5}{3}} \times a^{\frac{5}{3}} \times a^{\frac{5}{3}} = a^{\frac{5}{3}+\frac{5}{3}+\frac{5}{3}} = a^5.$$

That is, $a^{\frac{5}{3}}$ is such a quantity that when raised to the third power the result is a^5. Hence (Art. 198), $a^{\frac{5}{3}}$ must be the cube root of a^5; or, $a^{\frac{5}{3}} = \sqrt[3]{a^5}$.

We will now consider the general case:

Required the meaning of $a^{\frac{p}{q}}$, *where* p *and* q *are positive integers.*

By Rule I., $a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \cdots$ to q factors

$$= a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\cdots \text{to } q \text{ terms}} = a^{\frac{p}{q}\times q} = a^p.$$

That is, $a^{\frac{p}{q}}$ is such a quantity that when raised to the qth power the result is a^p. Therefore $a^{\frac{p}{q}}$ must be the qth root of a^p; or,

$$a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

Hence, in a fractional exponent, *the numerator denotes a power and the denominator a root.*

For example, $a^{\frac{3}{4}} = \sqrt[4]{a^3}$; $c^{\frac{5}{2}} = \sqrt{c^5}$; $x^{\frac{1}{3}} = \sqrt[3]{x}$; etc.

EXAMPLES.

219. Express the following with radical signs:

1. $a^{\frac{1}{4}}$. **3.** $2c^{\frac{1}{2}}$. **5.** $x^{\frac{3}{4}}y^{\frac{2}{3}}$. **7.** $4a^{\frac{m}{5}}b^{\frac{n}{6}}$. **9.** $5y^{\frac{4}{7}}z^{\frac{9}{2}}$.

2. $b^{\frac{3}{7}}$. **4.** $3am^{\frac{5}{4}}$. **6.** $m^{\frac{3}{5}}n^{\frac{5}{3}}$. **8.** $2c^{\frac{3}{8}}d^{\frac{2}{5}}$. **10.** $ab^{\frac{1}{3}}c^{\frac{4}{5}}d^{\frac{7}{2}}$.

Express the following with fractional exponents:

11. $\sqrt[5]{x^6}$. **13.** $\sqrt{n}$. **15.** $3\sqrt{m^5}$. **17.** $\sqrt[3]{a^4}\sqrt[5]{b}$.

12. $\sqrt[3]{y^2}$. **14.** $\sqrt[3]{c}$. **16.** $4\sqrt[7]{a^9}$. **18.** $\sqrt{x^5}\sqrt[5]{y^2}$.

19. $5\sqrt{m^r}\sqrt[3]{n^s}$. **20.** $2a\sqrt[n]{x}\sqrt[n]{y^m}$.

The value of a numerical quantity affected with a fractional exponent may be found by first extracting the root indicated by the denominator, and then raising the result to the power indicated by the numerator.

Thus, $\quad (-8)^{\frac{2}{3}} = (\sqrt[3]{-8})^2 = (-2)^2 = 4.$

Find the values of the following:

21. $9^{\frac{5}{2}}$. **23.** $36^{\frac{3}{2}}$. **25.** $(-27)^{\frac{5}{3}}$. **27.** $64^{\frac{7}{6}}$.

22. $27^{\frac{4}{3}}$. **24.** $16^{\frac{5}{4}}$. **26.** $(-32)^{\frac{4}{5}}$. **28.** $(-216)^{\frac{4}{3}}$.

220. *Required the meaning of* a^0.

Since Rule I. is to hold universally, we must have

$$a^m \times a^0 = a^{m+0} = a^m.$$

Therefore a^0 must be equal to 1.

That is, *any quantity whose exponent is* 0 *is equal to* 1.

221. We pass next to the case of negative exponents.

Required the meaning of a^{-3}.

By Rule I., $a^{-3} \times a^3 = a^{-3+3} = a^0 = 1.$ (Art. 220.)

Hence, $a^{-3} = \dfrac{1}{a^3}.$

We will now consider the general case:

Required the meaning of a^{-n}, *n being integral or fractional.*

By Rule I., $a^{-n} \times a^n = a^{-n+n} = a^0 = 1.$

Hence, $a^{-n} = \dfrac{1}{a^n}.$

For example,

$$a^{-2} = \frac{1}{a^2};\ a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}};\ 3x^{-1}y^{-\frac{1}{2}} = \frac{3}{xy^{\frac{1}{2}}};\ \text{etc.}$$

222. In connection with Art. 221, the following principle may be noticed:

Any factor of the numerator of a fraction may be transferred to the denominator, or any factor of the denominator to the numerator, if the sign of its exponent be changed.

Thus, the fraction $\frac{a^2b^3}{cd^4}$ can be written in any of the forms

$$\frac{b^3}{a^{-2}cd^4},\quad \frac{a^2b^3c^{-1}}{d^4},\quad \frac{1}{a^{-2}b^{-3}cd^4},\ \text{etc.}$$

EXAMPLES.

223. Write the following with positive exponents:

1. x^3y^{-5}.
2. $x^{-1}y^{\frac{2}{3}}$.
3. $m^2n^{-\frac{1}{2}}$.
4. $4xy^{-\frac{1}{3}}$.
5. $a^{-1}b^{-2}$.
6. $3a^{\frac{1}{2}}b^{-\frac{2}{5}}$.
7. $2x^{-4}y^{-\frac{3}{7}}$.
8. $a^{-5}b^{-2}c^3$.
9. $5a^{-3}b^{-2}c$.
10. $2m^{-6}n^{-4}$.
11. $3x^{-\frac{2}{3}}y^{-\frac{2}{7}}$.
12. $a^{-2}b^{-\frac{3}{4}}c^{-\frac{4}{3}}$.

Transfer the literal factors from the denominators to the numerators in the following:

13. $\frac{1}{x}$.
14. $\frac{a^2}{x^3}$.
15. $\frac{3}{x^{-2}}$.
16. $\frac{1}{2x^{\frac{3}{4}}}$.
17. $\frac{3c}{x^2y^{-1}}$.
18. $\frac{ab^2}{cd^{\frac{1}{2}}}$.
19. $\frac{5a^2}{2bc^3}$.
20. $\frac{a^3}{2x^{\frac{2}{3}}y^{\frac{3}{5}}}$.
21. $\frac{3x}{5m^{-4}n^{-\frac{2}{5}}}$.

Transfer the literal factors from the numerators to the denominators in the following:

22. $\frac{2x^2}{3}$.
23. $\frac{3x^{\frac{1}{2}}}{4a}$.
24. $\frac{x^{-3}}{2}$.
25. $\frac{2c^{-\frac{3}{4}}}{5}$.
26. $3a^{\frac{5}{4}}$.
27. $\frac{5a^{-2}c}{b^{\frac{1}{2}}}$.
28. $m^{-\frac{2}{3}}n^{\frac{1}{2}}$.
29. $\frac{x^{-1}y^{\frac{3}{5}}}{z^2}$.
30. $\frac{4a^{-2}b^{-\frac{3}{2}}}{3c^3}$.

224. Since the definitions of fractional and negative exponents were obtained on the supposition that Rule I., Art. 216, was to hold universally, we have for any values of m and n,

$$a^m \times a^n = a^{m+n}.$$

For example,

$$a^2 \times a^{-5} = a^{2-5} = a^{-3};$$

$$a^{\frac{3}{4}} \times a^{-\frac{2}{3}} = a^{\frac{3}{4}-\frac{2}{3}} = a^{\frac{1}{12}};$$

$$a \times a^{-\frac{5}{2}} = a^{1-\frac{5}{2}} = a^{-\frac{3}{2}};\ \text{etc.}$$

EXAMPLES.

Find the values of the following:

1. $a^3 \times a^{-1}$.
2. $a^2 \times a^{-2}$.
3. $x^{-1} \times x^{-5}$.
4. $n^{\frac{3}{4}} \times n^{-\frac{1}{3}}$.
5. $2x^{\frac{5}{2}} \times x^{-\frac{3}{2}}$.
6. $3a \times a^{-\frac{2}{3}}$.
7. $5c^{-3} \times 3c^{-\frac{1}{2}}$.
8. $a^3 \times \sqrt[3]{a^2}$.
9. $x^{-1} \times \sqrt[4]{x^{-3}}$.
10. $m^2 \times \dfrac{4}{\sqrt[5]{m}}$.
11. $2c^{-\frac{2}{7}} \times 3a\sqrt[5]{c^3}$.
12. $2a^{-3}b^{\frac{3}{4}} \times ab^{-1}$.
13. $x^2y^{-\frac{5}{3}} \times \dfrac{x^{-2}y^{\frac{8}{3}}}{2}$.
14. $\sqrt[6]{x} \times 5\sqrt{x^{-5}}$.
15. $\dfrac{1}{a^{\frac{1}{2}}b^{-2}} \times \dfrac{3}{a^{-3}b^{\frac{1}{3}}}$.

16. Multiply $a + 2a^{\frac{2}{3}} - 3a^{\frac{1}{3}}$ by $2 - 4a^{-\frac{1}{3}} - 6a^{-\frac{2}{3}}$.

$$\begin{array}{rrrr}
a + 2a^{\frac{2}{3}} & - 3a^{\frac{1}{3}} & & \\
2 - 4a^{-\frac{1}{3}} & - 6a^{-\frac{2}{3}} & & \\
\hline
2a + 4a^{\frac{2}{3}} & - 6a^{\frac{1}{3}} & & \\
- 4a^{\frac{2}{3}} & - 8a^{\frac{1}{3}} & + 12 & \\
 & - 6a^{\frac{1}{3}} & - 12 & + 18a^{-\frac{1}{3}} \\
\hline
2a & - 20a^{\frac{1}{3}} & & + 18a^{-\frac{1}{3}},\ \textit{Ans.}
\end{array}$$

Note. In examples like the above, it should be borne in mind that any quantity whose exponent is 0 is equal to 1. (Art. 220.)

Multiply the following:

17. $a^2 - 2 + a^{-2}$ by $a^2 + 2 + a^{-2}$.

18. $a^{\frac{4}{3}} + a^{\frac{2}{3}}x^{\frac{2}{3}} + x^{\frac{4}{3}}$ by $a^{\frac{2}{3}} - x^{\frac{2}{3}}$.

19. $x^{-\frac{3}{4}} - x^{-\frac{1}{2}} + x^{-\frac{1}{4}} - 1$ by $x^{-\frac{1}{4}} + 1$.

20. $x^{-2} - 2x^{-1} + 1 - 2x$ by $x^{-3} + 2x^{-2}$.

21. $3a^{-1} - a^{-2}b^{-1} + a^{-3}b^{-2}$ by $6a^3b^2 + 2a^2b + 2a$.

22. $2x^{\frac{2}{3}} - 3x^{\frac{1}{3}} - 4 + x^{-\frac{1}{3}}$ by $3x^{\frac{4}{3}} + x - 2x^{\frac{2}{3}}$.

23. $x^{-3}y^2 - x^{-2}y - 2x^{-1}$ by $2x^2y^{-1} + 2x^3y^{-2} - 4x^4y^{-3}$.

24. $a^{\frac{2}{3}}x^{-\frac{3}{4}} + 2 + a^{-\frac{2}{3}}x^{\frac{3}{4}}$ by $2a^{-\frac{2}{3}}x^{\frac{3}{4}} - 4a^{-\frac{4}{3}}x^{\frac{3}{2}} + 2a^{-2}x^{\frac{9}{4}}$.

25. $3a^{\frac{3}{4}}b^{-1} + a^{\frac{1}{4}} - 2a^{-\frac{1}{4}}b$ by $6a^{\frac{1}{4}}b^{-1} - 2a^{-\frac{1}{4}} - 3a^{-\frac{3}{4}}b$.

225. The rule of Art. 89 for the division of exponents holds universally; for, it follows from Art. 222 that

$$\frac{a^m}{a^n} = a^m \times a^{-n} = a^{m-n}. \qquad \text{(Art. 224.)}$$

For example, $\dfrac{a^{-\frac{3}{4}}}{a} = a^{-\frac{3}{4}-1} = a^{-\frac{7}{4}}$;

$$\frac{a^{\frac{1}{2}}}{a^{-2}} = a^{\frac{1}{2}+2} = a^{\frac{5}{2}};$$

$$\frac{a^{-3}}{a^{-\frac{2}{5}}} = a^{-3+\frac{2}{5}} = a^{-\frac{13}{5}}; \text{ etc.}$$

EXAMPLES.

Divide the following:

1. a^3 by a^{-1}.

2. a by a^3.

3. $a^{\frac{3}{7}}$ by $a^{\frac{4}{5}}$.

4. $a^{-\frac{1}{2}}$ by $a^{-\frac{4}{7}}$.

5. $3c^{-1}$ by $\sqrt[4]{c^5}$.

6. m^2 by $\sqrt[5]{m^{-2}}$.

7. $x^{\frac{1}{3}}$ by $\dfrac{1}{\sqrt[4]{x^3}}$.

8. $15a$ by $3a^{-1}\sqrt[3]{b}$.

9. $6x^{-1}y^{\frac{2}{3}}$ by $3xy^{-\frac{1}{5}}$.

10. Divide $2a^{\frac{2}{3}} - 20 + 18a^{-\frac{2}{3}}$ by $a + 2a^{\frac{2}{3}} - 3a^{\frac{1}{3}}$.

$$
\begin{array}{l|l}
2a^{\frac{2}{3}} - 20 + 18a^{-\frac{2}{3}} & a + 2a^{\frac{2}{3}} - 3a^{\frac{1}{3}} \\
\underline{2a^{\frac{2}{3}} + 4a^{\frac{1}{3}} - 6} & \overline{2a^{-\frac{1}{3}} - 4a^{-\frac{2}{3}} - 6a^{-1}}, \textit{ Ans.} \\
\quad -4a^{\frac{1}{3}} - 14 + 18a^{-\frac{2}{3}} & \\
\quad \underline{-4a^{\frac{1}{3}} - \ \ 8 + 12a^{-\frac{1}{3}}} & \\
\qquad\quad -6 - 12a^{-\frac{1}{3}} + 18a^{-\frac{2}{3}} & \\
\qquad\quad \underline{-6 - 12a^{-\frac{1}{3}} + 18a^{-\frac{2}{3}}} &
\end{array}
$$

Note. It is important to arrange the dividend and divisor in the same order of powers, and to keep this order throughout the work.

Divide the following:

11. $a - b$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$. **12.** $a^{-2} + 1$ by $a^{-\frac{2}{3}} + 1$.

13. $x^{-2} - 5x^{-1} - 46 - 40x$ by $x^{-1} + 4$.

14. $x^{-3} - 1$ by $x^{-\frac{9}{4}} - x^{-\frac{3}{2}} + x^{-\frac{3}{4}} - 1$.

15. $m - 3m^{\frac{2}{3}}n^{\frac{1}{3}} + 3m^{\frac{1}{3}}n^{\frac{2}{3}} - n$ by $m^{\frac{1}{3}} - n^{\frac{1}{3}}$.

16. $x^{-3}y^{-5} - 3x^{-5}y^{-7} + x^{-7}y^{-9}$ by $x^{-2}y^{-3} + x^{-3}y^{-4} - x^{-4}y^{-5}$.

17. $a + a^{\frac{1}{2}}b^{\frac{1}{2}} + b$ by $a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}}$.

18. $m^{-\frac{4}{3}} + m^{-\frac{2}{3}}n^{-2} + n^{-4}$ by $m^{-1} + m^{-\frac{2}{3}}n^{-1} + m^{-\frac{1}{3}}n^{-2}$.

226. We will now prove that Rule II., Art. 216, holds for all values of m and n.

We will consider three cases, in each of which m may have any value, positive or negative, integral or fractional.

Case I. Let n be a positive integer.

Then, from the definition of a positive integral exponent,

$$
\begin{aligned}
(a^m)^n &= a^m \times a^m \times a^m \times \cdots \text{ to } n \text{ factors} \\
&= a^{m+m+m+\cdots \text{ to } n \text{ terms}} = a^{mn}.
\end{aligned}
$$

CASE II. Let n be a positive fraction, which we will denote by $\frac{p}{q}$.

Then, $\quad (a^m)^{\frac{p}{q}} = \sqrt[q]{(a^m)^p}$, by the definition of Art. 218,

$= \sqrt[q]{a^{mp}}$, by Case I.,

$= a^{\frac{mp}{q}}$, by Art. 218,

$= a^{m \times \frac{p}{q}}$.

CASE III. Let n be a negative quantity, which we will denote by $-s$.

Then, $\quad (a^m)^{-s} = \dfrac{1}{(a^m)^s}$, by the definition of Art. 221,

$= \dfrac{1}{a^{ms}}$, by Cases I. or II.,

$= a^{-ms} = a^{m(-s)}$.

We have therefore for all values of m and n,

$$(a^m)^n = a^{mn}.$$

For example,

$$(a^{-\frac{2}{3}})^{\frac{1}{2}} = a^{-\frac{2}{3} \times \frac{1}{2}} = a^{-\frac{1}{3}};$$

$$(a^2)^{-3} = a^{2 \times -3} = a^{-6};$$

$$(a^{-3})^{-\frac{1}{3}} = a^{-3 \times -\frac{1}{3}} = a; \text{ etc.}$$

EXAMPLES.

227. Find the values of the following:

1. $(a^2)^{-3}$.	**5.** $(x^{-\frac{3}{4}})^{-2}$.	**9.** $(\sqrt[4]{m^3})^{\frac{2}{3}}$.	**13.** $\left(\dfrac{1}{\sqrt{c}}\right)^{\frac{2}{5}}$.
2. $(a^{-2})^2$.	**6.** $(a^{-1})^{\frac{1}{8}}$.	**10.** $(\sqrt[5]{y^{-3}})^{-5}$.	**14.** $\left(\dfrac{1}{\sqrt[4]{n^3}}\right)^{\frac{4}{3}}$.
3. $(a^3)^{\frac{5}{2}}$.	**7.** $(a^{\frac{1}{3}})^{\frac{6}{5}}$.	**11.** $\left(\dfrac{1}{a^2}\right)^{\frac{3}{2}}$.	**15.** $\sqrt[3]{[(x^{-\frac{1}{2}})^2]}$.
4. $(c^{-\frac{2}{5}})^{\frac{10}{3}}$.	**8.** $(\sqrt{x})^{-\frac{1}{3}}$.	**12.** $(x^{\frac{2n}{3}})^{-\frac{6}{n}}$.	**16.** $(a^{1-\frac{n}{m}})^{\frac{1}{m-n}}$.

228. *To prove that* $(ab)^n = a^n b^n$, *for any value of* n.

In Art. 193 we showed the truth of the theorem for a positive integral value of n.

Case I. Let n be a positive fraction, which we will denote by $\frac{p}{q}$.

By Art. 226, $\quad [(ab)^{\frac{p}{q}}]^q = (ab)^p.$

By Art. 193, $\quad [a^{\frac{p}{q}} b^{\frac{p}{q}}]^q = a^p b^p = (ab)^p.$

Therefore, $\quad [(ab)^{\frac{p}{q}}]^q = [a^{\frac{p}{q}} b^{\frac{p}{q}}]^q.$

Taking the qth root of both members,

$$(ab)^{\frac{p}{q}} = a^{\frac{p}{q}} b^{\frac{p}{q}}.$$

Case II. Let n be a negative quantity, which we will denote by $-s$.

Then, $\quad (ab)^{-s} = \dfrac{1}{(ab)^s} = \dfrac{1}{a^s b^s}$, by Art. 193, or Case I.

$$= a^{-s} b^{-s}.$$

MISCELLANEOUS EXAMPLES.

229. Square the following (Art. 95):

1. $a^{\frac{2}{3}} - b^{\frac{1}{2}}$. **2.** $a^{-\frac{3}{2}} + 2a$. **3.** $x^{-1}y^2 - 3x^2y^{-3}$.

Extract the square roots of the following:

4. $a^{-2}x^{\frac{3}{4}}$. **5.** $9mn^{\frac{1}{3}}$. **6.** $\dfrac{c^{\frac{2}{3}}d^{-\frac{5}{4}}}{4xy^3}$. **7.** $\dfrac{a^{-\frac{2}{3}}b^{-1}}{cd^4e^{\frac{1}{2}}}$.

8. $9x^{-4} - 12x^{-3} - 2x^{-2} + 4x^{-1} + 1$.

9. $4x^{\frac{4}{3}} + 4x^{\frac{5}{3}} - 15x^2 - 8x^{\frac{7}{3}} + 16x^{\frac{8}{3}}$.

10. $a^3b^{-\frac{2}{3}} - 4a^{\frac{3}{2}}b^{-\frac{1}{3}} + 6 - 4a^{-\frac{3}{2}}b^{\frac{1}{3}} + a^{-3}b^{\frac{2}{3}}$.

Extract the cube roots of the following:

11. ab^2. **12.** $-8x^{-4}y^{\frac{2}{3}}$. **13.** $27m^2n^{-\frac{2}{7}}$. **14.** $\dfrac{a^{-1}b}{64x^{\frac{1}{3}}}$.

15. $8y^{-2} - 12y^{-\frac{11}{6}} + 6y^{-\frac{5}{3}} - y^{-\frac{3}{2}}$.

16. $x^{\frac{1}{2}} - 9x^{\frac{2}{3}} + 33x^{\frac{5}{6}} - 63x + 66x^{\frac{7}{6}} - 36x^{\frac{4}{3}} + 8x^{\frac{3}{2}}$.

Reduce the following to their simplest forms:

17. $a^{x-y+2z}a^{2x+y-3z}a^z$.

18. $\dfrac{x^{m+n}x^{m+r}x^{r-m}}{x^{n+2m-r}}$.

19. $(x^a)^{-b} \div (x^{-a})^{-b}$.

20. $\left[x^{a^2-ab}x^{b^2-ab}\right]^{\frac{1}{a-b}}$.

21. $\left(\dfrac{a^{x+y}}{a^y}\right)^x \div \left(\dfrac{a^y}{a^{y-x}}\right)^{x-y}$.

22. $\left[\left(x^{\frac{1}{a-b}}\right)^{a-\frac{b^2}{a}}\right]^{\frac{a}{a+b}}$.

23. $\dfrac{x^{\frac{1}{2}}(a^{\frac{1}{2}} - x^{\frac{1}{2}}) - x^{\frac{3}{2}}(a^{-\frac{1}{2}} - x^{-\frac{1}{2}})}{2(a^{\frac{1}{2}} + x^{\frac{1}{2}})(a^{\frac{1}{2}} - x^{\frac{1}{2}})}$.

24. $\dfrac{a-b}{a^{\frac{1}{4}}b^{-\frac{1}{4}} + a^{-\frac{1}{4}}b^{\frac{1}{4}}}$.

25. $\dfrac{(1 - a^{\frac{1}{2}}x^{\frac{1}{2}})^2 + (x^{\frac{1}{2}} + a^{\frac{1}{2}})^2}{1 - a^{\frac{1}{2}}x^{\frac{1}{2}} + a^{\frac{1}{2}}(a^{\frac{1}{2}} + x^{\frac{1}{2}})}$.

26. $(4x^3 - 3x)(x^2+1)^{-\frac{1}{3}} + 3x(x^2+1)^{\frac{2}{3}}$.

27. $\dfrac{(1 - 3x + x^2)^{\frac{1}{2}} - x(x-3)(1 - 3x + x^2)^{-\frac{1}{2}}}{1 - 3x + x^2}$.

28. $\dfrac{m[x^{-\frac{1}{2}} + (m+x)^{-\frac{1}{2}}]}{2[x^{\frac{1}{2}} + (m+x)^{\frac{1}{2}}]} + \dfrac{m+2x}{2x^{\frac{1}{2}}(m+x)^{\frac{1}{2}}}$.

29. $\dfrac{x^2 + [1 + (1+x^2)^{\frac{1}{2}}]^2}{2[1 + (1+x^2)^{\frac{1}{2}}]}$.

XX. RADICALS.

230. A **Radical** is a root of a quantity indicated by a radical sign; as, $\sqrt{a}$, or $\sqrt[3]{x+1}$.

If the indicated root can be exactly obtained, it is called a *rational* quantity; if it cannot be exactly obtained, it is called an *irrational* or *surd* quantity.

231. The *degree* of a radical is denoted by the index of the radical sign; thus, $\sqrt[3]{x+1}$ is of the *third* degree.

232. *Similar Radicals* are those of the same degree, and with the same quantity under the radical sign; as, $2\sqrt[5]{ax}$ and $3\sqrt[5]{ax}$.

233. Most problems in radicals depend for their solution on the following important principle (Art. 228):

For any value of n, $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} \times b^{\frac{1}{n}}$.

That is, $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$.

TO REDUCE A RADICAL TO ITS SIMPLEST FORM.

234. A radical is in its *simplest form* when the quantity under the radical sign is not a perfect power of the degree denoted by any factor of the index of the radical, and has no factor which is a perfect power of the same degree as the radical.

Case I.

235. *When the quantity under the radical sign is a perfect power of the degree denoted by a factor of the index.*

1. Reduce $\sqrt[6]{8}$ to its simplest form.

$$\sqrt[6]{8} = \sqrt[6]{2^3} = 2^{\frac{3}{6}} = 2^{\frac{1}{2}} = \sqrt{2}, \textit{ Ans.}$$

EXAMPLES.

Reduce the following to their simplest forms:

2. $\sqrt[4]{25}$.	5. $\sqrt[12]{27}$.	8. $\sqrt[12]{64}$.	11. $\sqrt[4]{49m^4n^6}$.
3. $\sqrt[6]{9}$.	6. $\sqrt[10]{100}$.	9. $\sqrt[8]{25x^4}$.	12. $\sqrt[6]{125a^3b^9}$.
4. $\sqrt[9]{8}$.	7. $\sqrt[12]{81}$.	10. $\sqrt[10]{32a^5}$.	13. $\sqrt[mn]{a^nb^{2n}}$.

CASE II.

236. *When the quantity under the radical sign has a factor which is a perfect power of the same degree as the radical.*

1. Reduce $\sqrt[3]{54}$ to its simplest form.

$\sqrt[3]{54} = \sqrt[3]{27 \times 2} = \sqrt[3]{27} \times \sqrt[3]{2}$ (Art. 233) $= 3\sqrt[3]{2}$, *Ans.*

2. Reduce $\sqrt{18a^2b^5 - 27a^3b^4}$ to its simplest form.

$$\begin{aligned}\sqrt{18a^2b^5 - 27a^3b^4} &= \sqrt{9a^2b^4(2b - 3a)}\\ &= \sqrt{9a^2b^4} \times \sqrt{2b - 3a}\\ &= 3ab^2\sqrt{2b - 3a}, \textit{ Ans.}\end{aligned}$$

RULE.

Resolve the quantity under the radical sign into two factors, one of which is the highest perfect power of the same degree as the radical. Extract the required root of this factor, and prefix the result to the indicated root of the other.

Note. If the highest perfect power in the numerical portion of the quantity cannot be determined by inspection, it may be found by resolving the number into its prime factors.

Thus,
$$\begin{aligned}\sqrt{1944} = \sqrt{2^3 \times 3^5} &= \sqrt{2^2 \times 3^4} \times \sqrt{2 \times 3}\\ &= 2 \times 3^2 \times \sqrt{6} = 18\sqrt{6}.\end{aligned}$$

EXAMPLES.

Reduce the following to their simplest forms.

3. $\sqrt{50}$.
4. $3\sqrt{24}$.
5. $\sqrt{72}$.
6. $\sqrt[3]{320}$.
7. $2\sqrt[4]{80}$.
8. $\sqrt{98a^3b^2}$.
9. $\sqrt[3]{81x^4y^3}$.
10. $7\sqrt{63a^4b^5c^6}$.
11. $\sqrt[3]{250x^2y^3z^7}$.
12. $\sqrt{25x^3y^4-50x^4y^3}$.
13. $\sqrt[3]{54a^4b^5+135a^3b^4}$.
14. $\sqrt{(x^2-y^2)(x+y)}$.
15. $\sqrt{ax^2-6ax+9a}$.
16. $\sqrt{20x^2+60x+45}$.
17. $\sqrt{3m^3-54m^2n+243mn^2}$.

If the quantity under the radical sign is a fraction, *multiply both terms by such a quantity as will make the denominator a perfect power of the same degree as the radical.* Then proceed as before.

18. Reduce $\sqrt{\frac{9}{8a^3}}$ to its simplest form.

$$\sqrt{\frac{9}{8a^3}}=\sqrt{\frac{18a}{16a^4}}=\sqrt{\frac{9}{16a^4}\times 2a}=\frac{3}{4a^2}\sqrt{2a},\ \textit{Ans.}$$

Reduce the following to their simplest forms:

19. $\sqrt{\frac{3}{2}}$.
20. $\sqrt{\frac{5}{6}}$.
21. $\sqrt{\frac{7}{12}}$.
22. $\sqrt{\frac{4a^2}{27}}$.
23. $\sqrt[3]{\frac{3x}{4}}$.
24. $\sqrt[3]{\frac{5}{9}}$.
25. $\frac{3}{11}\sqrt{\frac{4}{7}}$.
26. $\sqrt{\frac{9a^2b^3}{10cd}}$.
27. $\sqrt{\frac{7xy^2}{8a^5}}$.
28. $\sqrt{\frac{ab^2}{4(a+x)}}$.
29. $\frac{a}{a^2-b^2}\sqrt{\frac{a^3c-2a^2bc+ab^2c}{b^3}}$.

237. *Conversely*, the coefficient of a radical may be introduced under the radical sign by raising it to the power denoted by the index.

1. Introduce the coefficient of $2a\sqrt[3]{3x^2}$ under the radical sign.

$$2a\sqrt[3]{3x^2} = \sqrt[3]{8a^3 \times 3x^2} = \sqrt[3]{24a^3x^2},\ \textit{Ans.}$$

Note. A rational quantity may be expressed in the form of a radical by raising it to the power denoted by the index, and writing the result under the corresponding radical sign.

EXAMPLES.

Introduce the coefficients of the following under the radical signs:

2. $3\sqrt{5}$. **4.** $3\sqrt[4]{2}$. **6.** $4\sqrt{5ab}$. **8.** $5a\sqrt[3]{2x^2}$.

3. $2\sqrt[3]{7}$. **5.** $4\sqrt[3]{5}$. **7.** $a^2b\sqrt[3]{ab^2}$. **9.** $3mn^3\sqrt[4]{\frac{mn^2}{27}}$.

10. $(x-1)\sqrt{\frac{x+1}{x-1}}$. **12.** $\frac{1+a}{1-a}\sqrt{\frac{1-a}{1+a}}$.

11. $(1+x)\sqrt{\frac{2}{1+x}-1}$. **13.** $\frac{2x^2-1}{x}\sqrt{\frac{1}{(2x^2-1)^2}-1}$.

ADDITION AND SUBTRACTION OF RADICALS.

238. The sum or difference of two similar radicals (Art. 232) may be found by prefixing the sum or difference of their coefficients to their common radical part.

1. Find the sum of $\sqrt{20}$ and $\sqrt{45}$.

By Art 236, $\sqrt{20} = \sqrt{4 \times 5} = 2\sqrt{5}$,

$\sqrt{45} = \sqrt{9 \times 5} = 3\sqrt{5}$.

Hence, $\sqrt{20} + \sqrt{45} = 2\sqrt{5} + 3\sqrt{5} = 5\sqrt{5}$, *Ans.*

2. Simplify $\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}-\sqrt{\frac{9}{8}}$.

$$\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}-\sqrt{\frac{9}{8}}=\sqrt{\frac{1}{4}\times 2}+\sqrt{\frac{1}{9}\times 6}-\sqrt{\frac{9}{16}\times 2}$$
$$=\frac{1}{2}\sqrt{2}+\frac{1}{3}\sqrt{6}-\frac{3}{4}\sqrt{2}$$
$$=\frac{1}{3}\sqrt{6}-\frac{1}{4}\sqrt{2},\ \textit{Ans.}$$

RULE.

Reduce each radical to its simplest form. Unite the similar radicals, and indicate the addition or subtraction of the dissimilar.

EXAMPLES.

Simplify the following:

3. $\sqrt{27}+\sqrt{12}$.

4. $\sqrt{96}+\sqrt{54}$.

5. $\sqrt{180}-\sqrt{45}$.

6. $\sqrt[3]{162}-\sqrt[3]{48}$.

7. $\sqrt{128}+\sqrt{98}+\sqrt{50}$.

8. $\sqrt{\frac{16}{15}}-\sqrt{\frac{3}{5}}$.

9. $\sqrt{4a^2b}+\sqrt{9b^3}$.

10. $\sqrt{75}+\sqrt{48}-\sqrt{245}$.

11. $\sqrt[3]{16}+\sqrt[3]{54}+\sqrt[3]{128}$.

12. $\sqrt{\frac{5}{9}}-\sqrt{\frac{1}{5}}+\sqrt{\frac{1}{45}}$.

13. $\sqrt{\frac{3}{8}}-\sqrt{\frac{1}{6}}+\sqrt{\frac{2}{27}}$.

14. $\sqrt[3]{\frac{1}{4}}+\sqrt[3]{\frac{1}{32}}+\sqrt[3]{\frac{2}{3}}$.

15. $7\sqrt{27}-\sqrt{75}-24\sqrt{\frac{1}{12}}-27\sqrt{\frac{1}{27}}$.

16. $\sqrt{27ab^2}+\sqrt{75a^3}+(a-3b)\sqrt{3a}$.

17. $\sqrt{9a^5+18a^4b}-\sqrt{4ab^6+8b^7}$.

18. $\sqrt[3]{24}+5\sqrt[3]{54}-\sqrt[3]{250}-\sqrt[3]{192}$.

19. $\sqrt{28a^2x-28ax+7x}-\sqrt{7a^2x+42ax+63x}$.

20. $x\sqrt{\dfrac{x-y}{x+y}}+y\sqrt{\dfrac{x+y}{x-y}}-\dfrac{3y^2-x^2}{x^2-y^2}\sqrt{x^2-y^2}$.

TO REDUCE RADICALS OF DIFFERENT DEGREES TO EQUIVALENT RADICALS OF THE SAME DEGREE.

239. 1. Reduce $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{5}$ to equivalent radicals of the same degree.

By Art 218, $\sqrt{2}=2^{\frac{1}{2}}=2^{\frac{6}{12}}=\sqrt[12]{2^6}=\sqrt[12]{64}$.

$$\sqrt[3]{3}=3^{\frac{1}{3}}=3^{\frac{4}{12}}=\sqrt[12]{3^4}=\sqrt[12]{81}.$$

$$\sqrt[4]{5}=5^{\frac{1}{4}}=5^{\frac{3}{12}}=\sqrt[12]{5^3}=\sqrt[12]{125}.$$

RULE.

Write the radicals with fractional exponents, and reduce these exponents to a common denominator.

Note. The relative magnitude of radicals may be compared by reducing them to the same degree. Thus, in Ex. 1, $\sqrt[12]{125}$ is greater than $\sqrt[12]{81}$, and $\sqrt[12]{81}$ than $\sqrt[12]{64}$. Hence, $\sqrt[4]{5}$ is greater than $\sqrt[3]{3}$, and $\sqrt[3]{3}$ than $\sqrt{2}$.

EXAMPLES.

Reduce the following to equivalent radicals of the same degree:

2. $\sqrt[3]{2}$ and $\sqrt{3}$.

3. $\sqrt{5}$, $\sqrt[3]{4}$, and $\sqrt[3]{3}$.

4. $\sqrt[3]{5}$, $\sqrt[4]{6}$, and $\sqrt[6]{7}$.

5. $\sqrt[3]{2a}$, $\sqrt[5]{3b}$, and $\sqrt[5]{4c}$.

6. $\sqrt[6]{xy}$, $\sqrt[4]{yz}$, and $\sqrt[8]{zx}$.

7. $\sqrt[6]{a+b}$ and $\sqrt[4]{a-b}$.

8. Which is the greater, $\sqrt[3]{2}$ or $\sqrt[5]{3}$?

9. Which is the greater, $\sqrt[4]{3}$ or $\sqrt[5]{5}$?

10. Arrange in order of magnitude $\sqrt{3}$, $\sqrt[3]{4}$, and $\sqrt[4]{7}$.

MULTIPLICATION OF RADICALS.

240. 1. Multiply $\sqrt{6}$ by $\sqrt{15}$.

By Art. 233,

$$\sqrt{6} \times \sqrt{15} = \sqrt{6 \times 15} = \sqrt{90} = 3\sqrt{10}, \textit{ Ans.}$$

2. Multiply $\sqrt{2a}$ by $\sqrt[3]{3a^2}$.

Reducing to equivalent radicals of the same degree,

$$\sqrt{2a} = (2a)^{\frac{1}{2}} = (2a)^{\frac{3}{6}} = \sqrt[6]{(2a)^3} = \sqrt[6]{8a^3}$$

$$\sqrt[3]{3a^2} = (3a^2)^{\frac{1}{3}} = (3a^2)^{\frac{2}{6}} = \sqrt[6]{(3a^2)^2} = \sqrt[6]{9a^4}$$

Hence, $\sqrt{2a} \times \sqrt[3]{3a^2} = \sqrt[6]{8a^3} \times \sqrt[6]{9a^4} = \sqrt[6]{72a^7}$

$$= a\sqrt[6]{72a}, \textit{ Ans.}$$

RULE.

Reduce the radicals to equivalent radicals of the same degree. Multiply together the quantities under the radical signs, and write the product under the common radical sign.

Note. The result should be reduced to its simplest form.

EXAMPLES.

Multiply the following:

3. $\sqrt{6}$ and $\sqrt{42}$.

4. $5\sqrt{10}$ and $3\sqrt{15}$.

5. $2\sqrt{3x}$ and $5\sqrt{15x}$.

6. $\sqrt[3]{a^2b}$ and $\sqrt[3]{abc^2}$.

7. $\frac{3}{4}\sqrt[3]{12}$ and $\frac{2}{3}\sqrt[3]{2}$.

8. $\sqrt[3]{2}$ and $\sqrt[4]{3}$.

9. $\sqrt{ax}$ and $\sqrt[3]{bx}$.

10. $\sqrt[3]{4a^2}$ and $\sqrt{2a}$.

11. $4\sqrt[5]{3}$ and $3\sqrt{2}$.

12. $\sqrt[4]{xy}$, $\sqrt[4]{yz}$, and $\sqrt[4]{zx}$.

13. $\sqrt{3}$, $\sqrt[3]{2}$, and $\sqrt[3]{\frac{1}{6}}$.

14. $\sqrt[5]{2x}$, $\sqrt[3]{3x}$, and $\sqrt[5]{\frac{1}{3x^2}}$.

15. Multiply $2\sqrt{3}+3\sqrt{2}$ by $3\sqrt{3}-\sqrt{2}$.

$$\begin{array}{l} 2\sqrt{3}+3\sqrt{2} \\ 3\sqrt{3}-\ \ \sqrt{2} \\ \hline \quad 18+9\sqrt{6} \\ \qquad\ -2\sqrt{6}-6 \\ \hline \quad 18+7\sqrt{6}-6=12+7\sqrt{6},\ \textit{Ans.} \end{array}$$

Note. It should be remembered that to multiply a radical of the second degree by itself simply removes the radical sign; thus, $\sqrt{3}\times\sqrt{3}=3$.

16. Multiply $3\sqrt{x^2+1}+4x$ by $2\sqrt{x^2+1}-x$.

$$\begin{array}{l} 3\sqrt{x^2+1}+4x \\ 2\sqrt{x^2+1}-\ \ x \\ \hline 6(x^2+1)+8x\sqrt{x^2+1} \\ \qquad\qquad -3x\sqrt{x^2+1}-4x^2 \\ \hline \quad 6x^2+6+5x\sqrt{x^2+1}-4x^2 \\ \qquad\qquad =2x^2+6+5x\sqrt{x^2+1},\ \textit{Ans.} \end{array}$$

Multiply the following:

17. $\sqrt{x}-2$ and $\sqrt{x}+3$.

18. $\sqrt{5}-3\sqrt{2}$ and $2\sqrt{5}+\sqrt{2}$.

19. $\sqrt{x}-4\sqrt{3}$ and $2\sqrt{x}+\sqrt{3}$.

20. $2\sqrt{a}-3\sqrt{b}$ and $4\sqrt{a}+\sqrt{b}$.

21. $\sqrt{x}-\sqrt{y}+\sqrt{z}$ and $\sqrt{x}+\sqrt{y}-\sqrt{z}$.

22. $\sqrt{x+1}-2\sqrt{x}$ and $2\sqrt{x+1}+\sqrt{x}$.

23. $\sqrt{2}-\sqrt{3}+\sqrt{5}$ and $\sqrt{2}+\sqrt{3}+\sqrt{5}$.

24. $3\sqrt{5}-2\sqrt{6}+\sqrt{7}$ and $6\sqrt{5}+4\sqrt{6}-2\sqrt{7}$.

25. $8\sqrt{3}+10\sqrt{2}-3\sqrt{5}$ and $4\sqrt{3}-5\sqrt{2}-\sqrt{5}$.

Expand the following (Art. 95):

26. $(2\sqrt{3}-3)^2$.

27. $(3\sqrt{8}+5\sqrt{3})^2$.

28. $(\sqrt{1-a^2}+a)^2$.

29. $(\sqrt{a+b}-\sqrt{a-b})^2$.

30. $(\sqrt{x^2+1}+x)(\sqrt{x^2+1}-x)$.

31. $(\sqrt{x+1}+\sqrt{x-1})(\sqrt{x+1}-\sqrt{x-1})$.

32. $(3\sqrt{2x+5}+2\sqrt{3x-1})(3\sqrt{2x+5}-2\sqrt{3x-1})$.

DIVISION OF RADICALS.

241. By Art. 233, $\sqrt[n]{ab}=\sqrt[n]{a}\times\sqrt[n]{b}$.

Whence, $$\frac{\sqrt[n]{ab}}{\sqrt[n]{a}}=\sqrt[n]{b}.$$

RULE.

Reduce the radicals to equivalent radicals of the same degree. Divide the quantities under the radical sign, and write the quotient under the common radical sign.

EXAMPLES.

1. Divide $\sqrt[3]{15}$ by $\sqrt{5}$.

Reducing to equivalent radicals of the same degree, we have

$$\frac{\sqrt[3]{15}}{\sqrt{5}}=\frac{\sqrt[6]{225}}{\sqrt[6]{125}}=\sqrt[6]{\frac{225}{125}}=\sqrt[6]{\frac{9}{5}},\ \textit{Ans.}$$

Divide the following:

2. $\sqrt{108}$ by $\sqrt{6}$.

3. $\sqrt{50c^3}$ by $\sqrt{2c}$.

4. $\sqrt[3]{9a^4}$ by $\sqrt[3]{3a}$.

5. $\sqrt{6}$ by $\sqrt[3]{3}$.

6. $\sqrt[3]{18}$ by $\sqrt{6}$.

7. $\sqrt[5]{2}$ by $\sqrt[4]{3}$.

8. $\sqrt[5]{12}$ by $\sqrt[3]{2}$.

9. $\sqrt[3]{4a}$ by $\sqrt[4]{2a}$.

10. $\sqrt[3]{3a^2b}$ by $\sqrt[5]{6a^3b^2}$.

11. $\sqrt[6]{12x^3y^2z^5}$ by $\sqrt[4]{2x^2yz^3}$.

INVOLUTION AND EVOLUTION OF RADICALS.

242. Any power or root of a radical may be found by using fractional exponents.

1. Raise $\sqrt[6]{12}$ to the third power.

$$(\sqrt[6]{12})^3 = (12^{\frac{1}{6}})^3 = 12^{\frac{3}{6}} = 12^{\frac{1}{2}} = \sqrt{12} = 2\sqrt{3},\ \textit{Ans.}$$

2. Raise $\sqrt[3]{2}$ to the fourth power.

$$(\sqrt[3]{2})^4 = (2^{\frac{1}{3}})^4 = 2^{\frac{4}{3}} = \sqrt[3]{2^4} = \sqrt[3]{16} = 2\sqrt[3]{2},\ \textit{Ans.}$$

Note 1. The following rule for the involution of radicals is evident from the above:

If possible, divide the index by the exponent of the required power; otherwise, raise the quantity under the radical sign to the required power.

EXAMPLES.

Find the values of the following:

3. $(\sqrt[5]{5})^3$.

4. $(\sqrt[4]{7})^2$.

5. $(\sqrt[3]{a^2x})^5$.

6. $(\sqrt[6]{18})^3$.

7. $(\sqrt[8]{a-b})^4$.

8. $(4\sqrt{3x})^3$.

9. $(\sqrt[12]{32})^3$.

10. $(3a\sqrt[3]{bx})^4$.

11. $(3\sqrt[6]{24a^4b^5})^2$.

12. Extract the cube root of $\sqrt{27x^3}$.

$$\sqrt[3]{(\sqrt{27x^3})} = (\sqrt{27x^3})^{\frac{1}{3}} = (\sqrt{(3x)^3})^{\frac{1}{3}} = [(3x)^{\frac{3}{2}}]^{\frac{1}{3}}$$
$$= (3x)^{\frac{1}{2}} = \sqrt{3x}, \textit{ Ans.}$$

13. Extract the square root of $\sqrt[3]{6}$.

$$\sqrt{(\sqrt[3]{6})} = (6^{\frac{1}{3}})^{\frac{1}{2}} = 6^{\frac{1}{6}} = \sqrt[6]{6}, \textit{ Ans.}$$

Note 2. The following rule for the evolution of radicals is evident from the above:

If possible, extract the required root of the quantity under the radical sign; otherwise, multiply the index of the radical by the index of the required root.

If the radical has a coefficient which is not a perfect power of the same degree as the required root, it should be introduced under the radical sign before applying the rule. Thus,

$$\sqrt[3]{2\sqrt{2}} = \sqrt[3]{\sqrt{8}} = \sqrt{2}.$$

Find the values of the following:

14. $\sqrt{(\sqrt{2})}$.

15. $\sqrt[3]{(\sqrt{125})}$.

16. $\sqrt[5]{(\sqrt{32})}$.

17. $\sqrt[3]{(\sqrt[5]{27a^3})}$.

18. $\sqrt[3]{(\sqrt[4]{a+b})}$.

19. $\sqrt{(\sqrt[3]{x^2-2x+1})}$.

20. $\sqrt[3]{(3\sqrt{3})}$.

21. $\sqrt[4]{(\sqrt[5]{x^8y^{12}})}$.

22. $\sqrt[5]{(4\sqrt{2})}$.

TO REDUCE A FRACTION HAVING AN IRRATIONAL DENOMINATOR TO AN EQUIVALENT FRACTION WHOSE DENOMINATOR IS RATIONAL.

CASE I.

243. *When the denominator is a monomial.*

The reduction is effected by multiplying both terms by a radical of the same degree as the denominator, having such a quantity under the radical sign as will make the denominator of the resulting fraction rational.

1. Reduce $\frac{5}{\sqrt[3]{3a^2}}$ to an equivalent fraction with a rational denominator.

Multiplying both terms by $\sqrt[3]{9a}$,

$$\frac{5}{\sqrt[3]{3a^2}} = \frac{5\sqrt[3]{9a}}{\sqrt[3]{3a^2}\sqrt[3]{9a}} = \frac{5\sqrt[3]{9a}}{\sqrt[3]{27a^3}} = \frac{5\sqrt[3]{9a}}{3a}, \textit{ Ans.}$$

EXAMPLES.

Reduce the following to equivalent fractions with rational denominators:

2. $\frac{3}{\sqrt{2}}$.

3. $\frac{1}{\sqrt[3]{4}}$.

4. $\frac{2x}{\sqrt{5xy}}$.

5. $\frac{5}{\sqrt[3]{9a^2}}$.

6. $\frac{1}{\sqrt[5]{16x^3}}$.

7. $\frac{2c}{\sqrt[4]{27a^2}}$.

Case II.

244. *When the denominator is a binomial, containing radicals of the second degree only.*

1. Reduce $\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$ to an equivalent fraction with a rational denominator.

Multiplying both terms by $\sqrt{5}-\sqrt{2}$,

$$\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}} = \frac{(\sqrt{5}-\sqrt{2})^2}{(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2})} = \frac{5-2\sqrt{10}+2}{5-2}$$

$$= \frac{7-2\sqrt{10}}{3}, \textit{ Ans.}$$

2. Reduce $\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}$ to an equivalent fraction with a rational denominator.

Multiplying both terms by $1+\sqrt{1-x}$,

$$\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}=\frac{(1+\sqrt{1-x})^2}{(1-\sqrt{1-x})(1+\sqrt{1-x})}$$

$$=\frac{1+2\sqrt{1-x}+1-x}{1-(1-x)}$$

$$=\frac{2-x+2\sqrt{1-x}}{x}, \textit{ Ans.}$$

RULE.

Multiply both terms of the fraction by the denominator with the sign between its terms changed.

EXAMPLES.

Reduce the following to equivalent fractions with rational denominators:

3. $\dfrac{4}{3+\sqrt{2}}$.

4. $\dfrac{4-\sqrt{3}}{2-\sqrt{3}}$.

5. $\dfrac{\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}}$.

6. $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$.

7. $\dfrac{2\sqrt{5}+\sqrt{2}}{\sqrt{5}-3\sqrt{2}}$.

8. $\dfrac{a-\sqrt{x}}{a+\sqrt{x}}$.

9. $\dfrac{\sqrt{a+1}-2}{\sqrt{a+1}-1}$.

10. $\dfrac{\sqrt{x+2}-\sqrt{x}}{\sqrt{x+2}+\sqrt{x}}$.

11. $\dfrac{a-\sqrt{a^2-1}}{a+\sqrt{a^2-1}}$.

12. $\dfrac{x+\sqrt{x^2-4}}{x-\sqrt{x^2-4}}$.

13. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$.

14. $\dfrac{\sqrt{a^2-1}-\sqrt{a^2+1}}{\sqrt{a^2-1}+\sqrt{a^2+1}}$.

245. The approximate value of a fraction, whose denominator is irrational, may be most conveniently found by reducing it to an equivalent fraction with a rational denominator.

1. Find the approximate value of $\frac{1}{2-\sqrt{2}}$ to three places of decimals.

$$\frac{1}{2-\sqrt{2}}=\frac{2+\sqrt{2}}{(2-\sqrt{2})(2+\sqrt{2})}$$

$$=\frac{2+\sqrt{2}}{4-2}$$

$$=\frac{2+1.414...}{2}=1.707..., \textit{ Ans.}$$

EXAMPLES.

Find the approximate values of the following to three decimal places:

2. $\frac{3}{\sqrt{2}-1}$. 3. $\frac{7}{\sqrt[3]{9}}$. 4. $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$. 5. $\frac{2\sqrt{5}-\sqrt{3}}{3\sqrt{5}+2\sqrt{3}}$.

IMAGINARY QUANTITIES.

246. An **Imaginary Quantity** is an indicated even root of a negative quantity (Art. 201); as, $\sqrt{-4}$, or $\sqrt[4]{-a^2}$.

In contradistinction, all other quantities, rational or irrational, are called *real* quantities.

247. Every imaginary square root can be expressed as the product of a real quantity multiplied by $\sqrt{-1}$. Thus,

$$\sqrt{-a^2}=\sqrt{a^2\times(-1)}=\sqrt{a^2}\times\sqrt{-1}=a\sqrt{-1};$$

$$\sqrt{-5}=\sqrt{5\times(-1)}=\sqrt{5}\sqrt{-1}; \text{ etc.}$$

248. Let it be required to find the powers of $\sqrt{-1}$.

By Art. 198, $\sqrt{-1}$ signifies a quantity which, when multiplied by itself, will produce -1; that is,

$$(\sqrt{-1})^2=-1.$$

Therefore,

$$(\sqrt{-1})^3=(\sqrt{-1})^2\times\sqrt{-1}=(-1)\times\sqrt{-1}=-\sqrt{-1};$$

$$(\sqrt{-1})^4=(\sqrt{-1})^2\times(\sqrt{-1})^2=(-1)\times(-1)=1;$$

$$(\sqrt{-1})^5=(\sqrt{-1})^4\times\sqrt{-1}=1\times\sqrt{-1}=\sqrt{-1}; \text{ etc.}$$

Thus the first four powers of $\sqrt{-1}$ are $\sqrt{-1}$, -1, $-\sqrt{-1}$, and 1; and for higher powers these terms recur in the same order.

MULTIPLICATION OF IMAGINARY QUANTITIES.

249. The product of two or more imaginary square roots may be found by aid of the principles of Arts. 247 and 248.

1. Multiply $\sqrt{-2}$ by $\sqrt{-3}$.

By Art. 247,

$$\begin{aligned}\sqrt{-2}\times\sqrt{-3}&=\sqrt{2}\sqrt{-1}\times\sqrt{3}\sqrt{-1}\\&=\sqrt{2}\sqrt{3}(\sqrt{-1})^2\\&=\sqrt{6}\times(-1)\ (\text{Art. } 248)=-\sqrt{6},\ \textit{Ans.}\end{aligned}$$

2. Multiply $\sqrt{-a^2}$, $\sqrt{-b^2}$, and $\sqrt{-c^2}$.

$$\begin{aligned}\sqrt{-a^2}\times\sqrt{-b^2}\times\sqrt{-c^2}&=a\sqrt{-1}\times b\sqrt{-1}\times c\sqrt{-1}\\&=abc(\sqrt{-1})^3=-abc\sqrt{-1},\ \textit{Ans.}\end{aligned}$$

RULE.

Reduce each imaginary quantity to the form of a real quantity multiplied by $\sqrt{-1}$. *Form the product of the real quantities, and multiply the result by the required power of* $\sqrt{-1}$.

EXAMPLES.

Multiply the following:

3. $4\sqrt{-3}$ and $2\sqrt{-2}$.

4. $\sqrt{-a^2}$ and $\sqrt{-x^2}$.

5. $-3\sqrt{-a}$ and $4\sqrt{-b}$.

6. $\sqrt{-3}$, $\sqrt{-4}$, and $\sqrt{-5}$.

7. $1-2\sqrt{-1}$ and $3+\sqrt{-1}$.

8. $4+\sqrt{-7}$ and $8-2\sqrt{-7}$.

9. $2\sqrt{-3}-3\sqrt{-2}$ and $4\sqrt{-3}+6\sqrt{-2}$.

10. $\sqrt{-1}$, $\sqrt{-9}$, $\sqrt{-16}$, and $\sqrt{-25}$.

Expand the following:

11. $(2-\sqrt{-3})^2$.

12. $(\sqrt{-3}+2\sqrt{-2})^2$.

13. $(1+\sqrt{-1})(1-\sqrt{-1})$.

14. $(a+\sqrt{-b})(a-\sqrt{-b})$.

15. $(x\sqrt{-x}+y\sqrt{-y})(x\sqrt{-x}-y\sqrt{-y})$.

16. $(1+\sqrt{-1})^2+(1-\sqrt{-1})^2$.

17. Divide $\sqrt{-x}$ by $\sqrt{-y}$.

$$\frac{\sqrt{-x}}{\sqrt{-y}}=\frac{\sqrt{x}\sqrt{-1}}{\sqrt{y}\sqrt{-1}}=\frac{\sqrt{x}}{\sqrt{y}}=\sqrt{\frac{x}{y}},\ \textit{Ans.}$$

Note. The rule of Art. 241 would have given the same result; hence, that rule applies to the division of all radicals, whether real or imaginary.

Divide the following:

18. $\sqrt{-6}$ by $\sqrt{-2}$.

19. $\sqrt{-24}$ by $\sqrt{-3}$.

20. $\sqrt[4]{-12}$ by $\sqrt[4]{-3}$.

21. $\sqrt[6]{-54}$ by $\sqrt[6]{-2}$.

PROPERTIES OF QUADRATIC SURDS.

250. A **Quadratic Surd** is the indicated square root of an imperfect square; as, $\sqrt{3}$, or $\sqrt{7}$.

251. *A quadratic surd cannot be equal to a rational quantity plus a quadratic surd.*

For, if possible, let $\sqrt{a} = b + \sqrt{c}$.

Squaring the equation, $a = b^2 + 2b\sqrt{c} + c$.

Or, $2b\sqrt{c} = a - b^2 - c$.

Whence, $\sqrt{c} = \dfrac{a - b^2 - c}{2b}$.

That is, a surd equal to a rational quantity, which is impossible. Hence $\sqrt{a}$ cannot be equal to $b + \sqrt{c}$.

252. *To prove that if $a + \sqrt{b} = c + \sqrt{d}$, then $a = c$, and $\sqrt{b} = \sqrt{d}$.*

If a is not equal to c, let $a = c + x$. Substituting, we have

$$c + x + \sqrt{b} = c + \sqrt{d}.$$

Or, $x + \sqrt{b} = \sqrt{d}$,

which is impossible by Art. 251. Hence $a = c$, and consequently $\sqrt{b} = \sqrt{d}$.

253. *To prove that if $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$, then $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$.*

Squaring the equation $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$,

we have $a + \sqrt{b} = x + 2\sqrt{xy} + y$.

Whence, by Art. 252, $a = x + y$, (1)

and $\sqrt{b} = 2\sqrt{xy}$. (2)

Subtracting (2) from (1), $a - \sqrt{b} = x - 2\sqrt{xy} + y$.

Extracting the square root, $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$.

SQUARE ROOT OF A BINOMIAL SURD.

254. The preceding principles serve to extract the square root of a binomial surd whose first term is rational.

For example, required the square root of $13-\sqrt{160}$.

Assume $\sqrt{13-\sqrt{160}}=\sqrt{x}-\sqrt{y}.$ (1)

Then, by Art. 253, $\sqrt{13+\sqrt{160}}=\sqrt{x}+\sqrt{y}.$ (2)

Multiplying (1) by (2), $\sqrt{169-160}=x-y.$

Or, $x-y=3.$ (3)

Squaring (1), $13-\sqrt{160}=x-2\sqrt{xy}+y.$

Whence, by Art. 252, $x+y=13.$ (4)

Adding (3) and (4), $2x=16$, or $x=8$.

Subtracting (3) from (4), $2y=10$, or $y=5$.

Substituting in (1), $\sqrt{13-\sqrt{160}}=\sqrt{8}-\sqrt{5}$

$=2\sqrt{2}-\sqrt{5}$, *Ans.*

255. Examples like the above may often be solved by inspection by expressing the given quantity in the form of a perfect trinomial square (Art. 108), as follows:

Reduce the surd term so that its coefficient may be 2. *Separate the rational term into two parts whose product is the quantity under the radical sign. Extract the square roots of these parts, and connect them by the sign of the surd term.*

1. Extract the square root of $8+\sqrt{48}$.

$$\sqrt{8+\sqrt{48}}=\sqrt{8+2\sqrt{12}}.$$

We then separate 8 into two parts whose product is 12. The parts are 6 and 2; hence,

$$\sqrt{8+2\sqrt{12}}=\sqrt{6+2\sqrt{6\times 2}+2}$$
$$=\sqrt{6}+\sqrt{2},\ \textit{Ans.}$$

2. Extract the square root of $22 - 3\sqrt{32}$.

$$\sqrt{22 - 3\sqrt{32}} = \sqrt{22 - \sqrt{288}} = \sqrt{22 - 2\sqrt{72}}.$$

We then separate 22 into two parts whose product is 72. The parts are 18 and 4; hence,

$$\sqrt{22 - 3\sqrt{32}} = \sqrt{18 - 2\sqrt{72} + 4}$$
$$= \sqrt{18} - \sqrt{4} = 3\sqrt{2} - 2, \textit{ Ans.}$$

EXAMPLES.

256. Extract the square roots of the following:

1. $12 + 2\sqrt{35}$.
2. $7 - 2\sqrt{12}$.
3. $9 + 2\sqrt{8}$.
4. $9 - 4\sqrt{5}$.
5. $16 + 6\sqrt{7}$.
6. $8 - \sqrt{60}$.
7. $15 + 4\sqrt{14}$.
8. $12 - \sqrt{108}$.
9. $20 - 5\sqrt{12}$.
10. $14 + 3\sqrt{20}$.
11. $23 + \sqrt{360}$.
12. $24 - 2\sqrt{63}$.
13. $33 + 20\sqrt{2}$.
14. $47 - 6\sqrt{10}$.
15. $67 - 7\sqrt{72}$.
16. $2m - 2\sqrt{m^2 - n^2}$.
17. $2a + x + 2\sqrt{a^2 + ax}$.

SOLUTION OF EQUATIONS CONTAINING RADICALS.

257. 1. Solve the equation $\sqrt{x^2 - 5} - x = -1$.

Transposing, $\sqrt{x^2 - 5} = x - 1$.

Squaring both members, $x^2 - 5 = x^2 - 2x + 1$.

Transposing and uniting terms, $2x = 6$.

$x = 3$, *Ans.*

2. Solve the equation $\sqrt{2x-1}+\sqrt{2x+6}=7$.

Transposing $\sqrt{2x-1}$,

$$\sqrt{2x+6}=7-\sqrt{2x-1}.$$

Squaring, $$2x+6=49-14\sqrt{2x-1}+2x-1.$$

Transposing and uniting,

$$14\sqrt{2x-1}=42.$$

Or, $$\sqrt{2x-1}=3.$$

Squaring, $$2x-1=9.$$

$$2x=10.$$

$$x=5, \textit{ Ans.}$$

RULE.

Transpose the terms of the equation so that a radical term may stand alone in one member; then raise both members to a power of the same degree as the radical.

If there are still radical terms remaining, repeat the operation.

Note. The equation should be simplified as much as possible before performing the involution.

EXAMPLES.

3. $\sqrt{5x-1}-2=1$.

4. $5=\sqrt[3]{2x}+3$.

5. $\sqrt[3]{4x+3}=3$.

6. $\sqrt{4x^2-19}-2x=-1$.

7. $\sqrt{x^2-3x+6}=2-x$.

8. $\sqrt[3]{x^3-6x^2}-x+2=0$.

9. $\sqrt{x}+\sqrt{x+5}=5$.

10. $\sqrt{x-32}+\sqrt{x}=16$.

11. $\sqrt{x-3}-\sqrt{x+12}=-3$.

12. $\sqrt{2x-7}+\sqrt{2x+9}=8$.

13. $\sqrt{3x+10} - \sqrt{3x+25} = -3.$

14. $\sqrt{(x-a)^2 + 2ab + b^2} = x - a + b.$

15. $\sqrt{x^2 - 3x + 5} - \sqrt{x^2 - 5x - 2} = 1.$

16. $\sqrt{x} - \sqrt{x-3} = \dfrac{2}{\sqrt{x}}.$

17. $\sqrt{x-1} + \sqrt{x+4} = \sqrt{4x+5}.$

18. $\sqrt{x^2 + 4x + 12} + \sqrt{x^2 - 12x - 20} = 8.$

19. $\dfrac{\sqrt{x}-3}{\sqrt{x}+7} = \dfrac{\sqrt{x}-4}{\sqrt{x}+1}.$

20. $\sqrt{3x} + \sqrt{3x+13} = \dfrac{91}{\sqrt{3x+13}}.$

21. $\sqrt{x+1} + \sqrt{x-2} - \sqrt{4x-3} = 0.$

22. $\sqrt{x} + \sqrt{x+a} = \dfrac{2a}{\sqrt{x+a}}.$

23. $\sqrt{\{9 + x\sqrt{x^2-3}\}} = x - 3.$

24. $\dfrac{a}{\sqrt{a-x}} - \dfrac{x}{\sqrt{b-x}} = \sqrt{b-x}.$

25. $\sqrt{x+a} + \sqrt{x+b} = \sqrt{4x+a+3b}.$

26. $\sqrt{\{1 + x\sqrt{x^2+16}\}} = x + 1.$

27. $\sqrt{\{a^2 - 2ax + x^2\sqrt{3a-x}\}} = a - x.$

XXI. QUADRATIC EQUATIONS.

258. A **Quadratic Equation,** or an equation of the *second degree* (Art. 167), is one in which the *square* is the highest power of the unknown quantity.

A **Pure Quadratic Equation** is one which contains only the square of the unknown quantity; as, $ax^2 = b$.

An **Affected Quadratic Equation** is one which contains both the square and the first power of the unknown quantity; as, $ax^2 + bx + c = 0$.

PURE QUADRATIC EQUATIONS.

259. A pure quadratic equation is solved by reducing it to the form $x^2 = a$, and then extracting the square roots of both members.

1. Solve the equation $3x^2 + 7 = \dfrac{5x^2}{4} + 35$.

Clearing of fractions, $12x^2 + 28 = 5x^2 + 140$.

Transposing and uniting, $7x^2 = 112$.

Or, $x^2 = 16$.

Taking the square root of both members,

$$x = \pm 4, \ \textit{Ans.}$$

Note 1. The double sign is placed before the result because the square root of a number is either positive or negative (Art. 201).

2. Solve the equation $7x^2 - 5 = 5x^2 - 13$.

Transposing and uniting, $2x^2 = -8$.

Or, $x^2 = -4$.

Whence,
$$\begin{aligned} x &= \pm\sqrt{-4} \\ &= \pm 2\sqrt{-1}, \ \textit{Ans.} \end{aligned}$$

Note 2. Since the square root of a negative quantity is imaginary (Art. 246), the values of x can only be indicated.

EXAMPLES.

Solve the following equations:

3. $4x^2 - 7 = 29.$

4. $5x^2 + 5 = 3x^2 + 55.$

5. $\dfrac{5}{6x^2} - \dfrac{7}{4x^2} = -\dfrac{33}{16}.$

6. $4 - \sqrt{3x^2 + 16} = 6.$

7. $ax^2 + b = c.$

8. $\dfrac{5}{4-x} = \dfrac{8}{3} - \dfrac{5}{4+x}.$

9 $2(x+3)(x-3) = (x+1)^2 - 2x.$

10. $(3x-2)(2x+5) + (5x+1)(4x-3) - 91 = 0.$

11. $\dfrac{x^2}{2} - 3 + \dfrac{5x^2}{12} = \dfrac{7}{24} - x^2 + \dfrac{335}{24}.$

12. $\dfrac{2x^2-5}{3} - \dfrac{3x^2+2}{7} - \dfrac{x^2-10}{6} = 0.$

13. $\dfrac{a}{x^2-b} = \dfrac{b}{x^2-a}.$

14. $\dfrac{4x^2-3}{2x^2-1} = \dfrac{2(9x^2+2)}{3(3x^2+2)}.$

15. $(2x-a)(x-b) + (2x+a)(x+b) = a^2 + b^2.$

16. $\dfrac{5x^2-1}{x^2-3} - \dfrac{3x^2+1}{x^2+2} - \dfrac{89}{(x^2-3)(x^2+2)} = 2.$

17. $x + \sqrt{x^2+3} = \dfrac{6}{\sqrt{x^2+3}}.$

18. $\dfrac{1}{1-\sqrt{1-x^2}} - \dfrac{1}{1+\sqrt{1-x^2}} = \dfrac{\sqrt{3}}{x^2}.$

AFFECTED QUADRATIC EQUATIONS.

260. An affected quadratic equation is solved by adding to both members such a quantity as will make the first member a perfect square; an operation which is termed *completing the square.*

FIRST METHOD OF COMPLETING THE SQUARE.

261. Every affected quadratic equation can be reduced to the form

$$x^2+px=q;$$

where p and q represent any quantities whatever, positive or negative, integral or fractional.

Let it be required to solve the equation $x^2+3x=4$.

In any trinomial square (Art. 108), the middle term is twice the product of the square roots of the first and third terms; hence the square root of the third term is equal to *the second term divided by twice the square root of the first.*

Therefore the *square root* of the quantity which must be added to x^2+3x to make it a perfect square, is $\frac{3x}{2x}$, or $\frac{3}{2}$.

Adding to both members the square of $\frac{3}{2}$, or $\frac{9}{4}$, we have

$$x^2+3x+\frac{9}{4}=4+\frac{9}{4}=\frac{25}{4}.$$

Extracting the square root of both members,

$$x+\frac{3}{2}=\pm\frac{5}{2}.$$

Transposing $\frac{3}{2}$, $\qquad x=-\frac{3}{2}+\frac{5}{2}$, or $-\frac{3}{2}-\frac{5}{2}$.

Whence, $\qquad x=1$ or -4, *Ans.*

262. From the above operation we derive the following rule:

Reduce the equation to the form $x^2+px=q$.

Complete the square by adding to both members the square of half the coefficient of x.

Extract the square root of both members, and solve the simple equation thus formed.

1. Solve the equation $3x^2 - 8x = -4$.

Dividing by 3, $x^2 - \frac{8x}{3} = -\frac{4}{3}$,

which is in the form $x^2 + px = q$.

Adding to both members the square of $\frac{4}{3}$, or $\frac{16}{9}$,

$$x^2 - \frac{8x}{3} + \frac{16}{9} = -\frac{4}{3} + \frac{16}{9} = \frac{4}{9}.$$

Extracting the square root,

$$x - \frac{4}{3} = \pm\frac{2}{3}.$$

Whence, $x = \frac{4}{3} \pm \frac{2}{3} = 2$ or $\frac{2}{3}$, *Ans.*

Note. These values may be verified as follows:

Putting $x = 2$ in the given equation, $12 - 16 = -4$.

Putting $x = \frac{2}{3}$, $\frac{4}{3} - \frac{16}{3} = -4$.

If the coefficient of x^2 is negative, it is necessary to change the sign of each term.

2. Solve the equation $-3x^2 - 7x = \frac{10}{3}$.

Dividing by -3, $x^2 + \frac{7x}{3} = -\frac{10}{9}$.

Adding to both members the square of $\frac{7}{6}$, or $\frac{49}{36}$,

$$x^2 + \frac{7x}{3} + \frac{49}{36} = -\frac{10}{9} + \frac{49}{36} = \frac{9}{36}.$$

Extracting the square root,

$$x + \frac{7}{6} = \pm\frac{3}{6}.$$

Whence, $x = -\frac{7}{6} \pm \frac{3}{6} = -\frac{2}{3}$ or $-\frac{5}{3}$, *Ans.*

EXAMPLES.

Solve the following equations:

3. $x^2 + 4x = 5$.
4. $x^2 - 5x = -4$.
5. $x^2 - 7x = -12$.
6. $x^2 + x = 6$.
7. $3x^2 - 4x = 4$.
8. $2x^2 + 5x = -2$.
9. $4x^2 - 8x + 3 = 0$.
10. $4x^2 - 3 = 11x$.
11. $3 - x - 2x^2 = 0$.
12. $14 + 15x - 9x^2 = 0$.

263. If the coefficient of x^2 is a perfect square, it is convenient to complete the square directly by the principle of Art. 261; that is, *by adding to both members the square of the quotient obtained by dividing the second term by twice the square root of the first.*

1. Solve the equation $9x^2 - 5x = 4$.

The quotient of the second term divided by twice the square root of the first, is $\frac{5}{6}$. Adding the square of $\frac{5}{6}$ to both members,

$$9x^2 - 5x + \frac{25}{36} = 4 + \frac{25}{36} = \frac{169}{36}.$$

Extracting the square root,

$$3x - \frac{5}{6} = \pm\frac{13}{6}.$$

$$3x = \frac{5}{6} \pm \frac{13}{6} = 3 \text{ or } -\frac{4}{3}.$$

Whence, $x = 1$ or $-\frac{4}{9}$, *Ans.*

Note. If the coefficient of x^2 is not a perfect square, it may be made so by multiplication.

Thus, in the equation $18x^2 + 5x = 2$, the coefficient of x^2 may be made a perfect square by multiplying each term by 2.

If the coefficient of x^2 is negative, the sign of each term must be changed.

EXAMPLES.

Solve the following equations:

2. $4x^2 + 3x = 10$.
3. $9x^2 + 2x = 11$.
4. $25x^2 - 15x = -2$.
5. $4x^2 - 7x = -3$.
6. $2x^2 + 15x = -13$.
7. $8x^2 + x - 34 = 0$.
8. $11x + 12 - 36x^2 = 0$.
9. $6x^2 - 5x = -1$.
10. $32x^2 + 20x - 7 = 0$.
11. $48x^2 - 32x = 3$.

SECOND METHOD OF COMPLETING THE SQUARE.

264. Every affected quadratic can be reduced to the form

$$ax^2 + bx = c.$$

Multiplying each term by $4a$, we have

$$4a^2x^2 + 4abx = 4ac.$$

Completing the square by adding to both members the square of b (Art. 263),

$$4a^2x^2 + 4abx + b^2 = b^2 + 4ac.$$

Extracting the square root,

$$2ax + b = \pm\sqrt{b^2 + 4ac}.$$

Transposing, $$2ax = -b \pm \sqrt{b^2 + 4ac}.$$

Dividing by $2a$, $$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}.$$

265. From the above operation we derive the following rule:

Reduce the equation to the form $ax^2 + bx = c$.

Multiply both members by four times the coefficient of x^2, and add to each the square of the coefficient of x in the given equation.

Extract the square root of both members, and solve the simple equation thus formed.

Note. The advantage of this method over the preceding is in avoiding fractions in completing the square.

1. Solve the equation $2x^2 - 7x = -3$.

Multiplying both members by 4 times 2, or 8,

$$16x^2 - 56x = -24.$$

Adding to each member the square of 7,

$$16x^2 - 56x + 49 = -24 + 49 = 25.$$

Extracting the square root,

$$4x - 7 = \pm 5.$$

Transposing, $\quad 4x = 7 \pm 5 = 12 \text{ or } 2.$

Dividing by 4, $\quad x = 3 \text{ or } \frac{1}{2}$, *Ans.*

If the coefficient of x in the given equation is an *even* number, fractions may be avoided, and the rule modified, as follows:

Multiply both members by the coefficient of x^2, and add to each the square of half the coefficient of x in the given equation.

2. Solve the equation $7x^2 + 4x = 51$.

Multiplying both members by 7,

$$49x^2 + 28x = 357.$$

Adding to each member the square of 2,

$$49x^2 + 28x + 4 = 361.$$

Extracting the square root,

$$7x + 2 = \pm 19.$$

$$7x = -2 \pm 19 = 17 \text{ or } -21.$$

Whence, $\quad x = \frac{17}{7} \text{ or } -3$, *Ans.*

EXAMPLES.

Solve the following equations:

3. $2x^2 + 5x = 3.$
4. $4x^2 - x = 3.$
5. $x^2 - 3x = 18.$
6. $3x^2 + 4x = 4.$
7. $8x^2 + 2x = 3.$
8. $2x^2 - 7x = 15.$
9. $7x^2 - 16x + 4 = 0.$
10. $17x + 20 = -3x^2.$
11. $5x^2 - 3 = 14x.$
12. $2 + x - 6x^2 = 0.$
13. $8x^2 + 6x + 1 = 0.$
14. $7x + 3 = 6x^2.$
15. $15x^2 - 8x = -1.$
16. $41x - 14 - 15x^2 = 0.$

MISCELLANEOUS EXAMPLES.

266. The following equations may be solved by either of the preceding methods; preference being given to the one best adapted to the example considered.

1. $\frac{x^2}{2} + \frac{x}{3} + \frac{1}{24} = 0.$
2. $\frac{2}{x} + \frac{x}{2} = -\frac{5}{2}.$
3. $\frac{1}{2x} = \frac{7}{6x^2} - \frac{2}{3}.$
4. $\frac{2}{5} - \frac{5}{2x} = -\frac{15}{4x^2}.$
5. $(x+5)(x-5) - (11x+1) = 0.$
6. $4x(18x-1) = (10x-1)^2.$
7. $(3x-5)^2 - (x+2)^2 = -5.$
8. $(x+3)^3 - (x-1)^3 = 19.$
9. $(x-1)^2 - (3x+8)^2 - (2x+5)^2 = 0.$
10. $\frac{2x+3}{8+x} - \frac{2x+9}{3x+4} = 0.$
11. $\frac{5}{x} - \frac{3x+1}{x^2} = \frac{1}{4}.$
12. $4x - \frac{14-x}{x+1} = 14.$
13. $\frac{21}{5-x} - \frac{x}{7} = \frac{25}{7}.$

14. $\frac{3x^2}{x-7}-\frac{1-8x}{10}=\frac{x}{5}.$

15. $\frac{x}{x-1}-\frac{x-1}{x}=\frac{3}{2}.$

16. $\frac{x}{5-x}-\frac{5-x}{x}=\frac{15}{4}.$

17. $\frac{x+1}{x+2}-\frac{x+3}{x+4}=\frac{8}{3}.$

18. $\sqrt{20+x-x^2}=2x-10.$

19. $2\sqrt{x}+\frac{2}{\sqrt{x}}=5.$

20. $\frac{2x-1}{x}-\frac{3x}{3x-1}+\frac{1}{2}=0.$

21. $\frac{x^3-x^2+7}{x^2+3x-1}=x+\frac{11}{3}.$

22. $\frac{2x^2+3x-5}{3x^2+4x-1}=\frac{2x^2-x-1}{3x^2-2x+7}.$

23. $\frac{7}{x^2-4}-\frac{3}{x+2}=\frac{22}{5}.$

24. $\frac{1}{x^2-1}+\frac{1}{3}=\frac{1}{3(x-1)}+\frac{1}{x+1}.$

25. $\frac{x+3}{x+2}+\frac{x-3}{x-2}=\frac{2x-3}{x-1}.$

26. $\frac{12+5x}{12-5x}+\frac{2+x}{x}=\frac{1}{1-5x}.$

27. $\sqrt{4x-3}-\sqrt{x+1}=1.$

28. $2\sqrt{x}=\sqrt{x+5}+\frac{3}{\sqrt{x+5}}.$

29. $\frac{x+2}{x-1}=\frac{2x+16}{x+5}-\frac{x-2}{x+1}.$

30. $\sqrt{3x+1}+\sqrt{2x-1}=\sqrt{9x+4}.$

The same methods are applicable to the solution of literal quadratic equations.

31. Solve the equation $x^2 - 2mx = 2m + 1$.

Completing the square by adding m^2 to both members,

$$x^2 - 2mx + m^2 = m^2 + 2m + 1 = (m + 1)^2.$$

Extracting the square root,

$$x - m = \pm (m + 1).$$

Whence,

$$x = m + (m+1), \text{ or } m - (m+1)$$
$$= 2m + 1 \text{ or } -1, \textit{ Ans.}$$

32. Solve the equation $x^2 + ax - bx - ab = 0$.

The equation may be written,

$$x^2 + (a - b)x = ab.$$

Multiplying both members by 4 times the coefficient of x^2,

$$4x^2 + 4(a - b)x = 4ab.$$

Adding to each member the square of $a - b$,

$$4x^2 + 4(a - b)x + (a - b)^2 = (a - b)^2 + 4ab$$
$$= a^2 + 2ab + b^2.$$

Extracting the square root,

$$2x + (a - b) = \pm (a + b).$$

Whence,

$$2x = -(a - b) \pm (a + b).$$

Hence,

$$2x = -a + b + a + b = 2b,$$

or,

$$2x = -a + b - a - b = -2a.$$

Dividing by 2,

$$x = -a \text{ or } b, \textit{ Ans.}$$

Note. If several terms contain the same power of x, the coefficient of that power should be placed in a parenthesis, as shown in Ex. 32.

Solve the following equations:

33. $x^2 - 2ax = (b + a)(b - a)$.

34. $x^2 - ax + bx = ab$.

35. $x^2 - (a + 1)x = -a$.

36. $x^2+2(c+8)x=-32c.$

37. $x^2-m^2(1-m)x=m^5.$

38. $acx^2-bcx-adx=-bd.$

39. $(x+2p)^3=(x+p)^3+37p^3.$

40. $6x^2+9ax+2bx=-3ab.$

41. $\dfrac{2x(a-x)}{3a-2x}=\dfrac{a}{4}.$

42. $\dfrac{x^2}{x+1}=\dfrac{m^2}{m+1}.$

43. $x+\dfrac{1}{x}=\dfrac{a}{b}+\dfrac{b}{a}.$

44. $\dfrac{x+1}{\sqrt{x}}=\dfrac{a+1}{\sqrt{a}}.$

45. $\sqrt{(a+b)x-4ab}=x-2b.$

46. $\sqrt{x-4ab}=\dfrac{(a+b)(a-b)}{\sqrt{x}}.$

47. $2\sqrt{x-m}+3\sqrt{2x}=\dfrac{7m+5x}{\sqrt{x-m}}.$

48. $\dfrac{1}{a+\sqrt{a^2-x}}+\dfrac{1}{a-\sqrt{a^2-x}}=1+\dfrac{x}{a}.$

49. $\sqrt{x+a}+\sqrt{x+2a}=\sqrt{2x+3a}.$

50. $\dfrac{x^2+1}{x}=\dfrac{a+b}{c}+\dfrac{c}{a+b}.$

51. $\dfrac{1}{a+b+x}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{x}.$

52. $x^2+bx+cx=(a+c)(a-b).$

53. $abx^2+\dfrac{3a^2x}{c}=\dfrac{6a^2+ab-2b^2}{c^2}-\dfrac{b^2x}{c}.$

54. $(3a^2+b^2)(x^2-x+1)=(3b^2+a^2)(x^2+x+1).$

SOLUTION OF QUADRATIC EQUATIONS BY A FORMULA.

267. It was shown in Art. 264 that if $ax^2 + bx = c$, then

$$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}. \qquad (1)$$

This result may be used as a formula for the solution of quadratic equations, as follows:

1. Solve the equation $2x^2 + 5x = 18$.

In this case, $a = 2$, $b = 5$, $c = 18$; substituting these values in (1),

$$x = \frac{-5 \pm \sqrt{25 + 144}}{4} = \frac{-5 \pm \sqrt{169}}{4}$$

$$= \frac{-5 \pm 13}{4} = 2 \text{ or } -\frac{9}{2}, \textit{ Ans.}$$

2. Solve the equation $110x^2 - 21x = -1$.

In this case, $a = 110$, $b = -21$, $c = -1$; therefore,

$$x = \frac{21 \pm \sqrt{441 - 440}}{220} = \frac{21 \pm 1}{220} = \frac{1}{10} \text{ or } \frac{1}{11}, \textit{ Ans.}$$

Note. Particular attention must be paid to the *signs* of the coefficients in substituting.

EXAMPLES.

Solve the following equations:

3. $2x^2 + 5x = 18$.
4. $3x^2 - 2x = 5$.
5. $x^2 - 7x = -10$.
6. $5x^2 + x = 18$.
7. $6x^2 + 7x = -1$.
8. $5x^2 - 11x = -2$.
9. $4x^2 - 8x - 5 = 0$.
10. $6x^2 + 25x + 14 = 0$.
11. $30x - 16 = 9x^2$.
12. $27 + 39x - 10x^2 = 0$.

XXII. PROBLEMS.

INVOLVING QUADRATIC EQUATIONS.

268. **1.** A man sold a watch for \$21, and lost as much per cent as the watch cost him. Required the cost of the watch.

Let $x =$ the cost in dollars.

Then, $x =$ the loss per cent,

and $x \times \frac{x}{100} = \frac{x^2}{100} =$ the loss in dollars.

By the conditions, $\frac{x^2}{100} = x - 21.$

Solving this equation, $x = 70$ or 30.

That is, the cost of the watch was either \$70 or \$30; for each of these values satisfies the given conditions.

2. A farmer bought some sheep for \$72. If he had bought 6 more for the same money, they would have cost \$1 apiece less. How many did he buy?

Let $x =$ the number bought.

Then, $\frac{72}{x} =$ the price paid for one,

and $\frac{72}{x+6} =$ the price if there had been 6 more.

By the conditions, $\frac{72}{x} = \frac{72}{x+6} + 1.$

Solving, $x = 18$ or -24.

Only the positive value of x is admissible, as the negative value does not answer to the conditions of the problem. The number of sheep, therefore, was 18.

Note 1. In solving problems which involve quadratics, there will always be two values of the unknown quantity; but only those values should be retained as answers which satisfy the conditions of the problem.

Note 2. If, in the given problem, the words "6 more" had been changed to "6 *fewer*," and "$1 apiece less" to "$1 apiece *more*," we should have found the answer 24.

In many cases where the solution of a problem gives a negative result, the wording may be changed so as to form an analogous problem to which the absolute value of the negative result is an answer.

PROBLEMS.

3. I bought a lot of flour for $175; and the number of dollars per barrel was $\frac{7}{4}$ of the number of barrels. How many barrels were purchased, and at what price?

4. Separate the number 15 into two parts the sum of whose squares shall be 117.

5. Find two numbers whose product is 126, and quotient $3\frac{1}{2}$.

6. I have a rectangular field of corn containing 6250 hills. The number of hills in the length exceeds the number in the breadth by 75. How many hills are there in the length, and in the breadth?

7. Find two numbers whose difference is 9, and whose sum multiplied by the greater is 266.

8. The sum of the squares of two consecutive numbers is 113. What are the numbers?

9. A man cut two piles of wood, whose united contents were 26 cords, for $35.60. The labor on each cost as many dimes per cord as there were cords in the pile. Required the number of cords in each pile.

10. Find two numbers whose sum is 8, and the sum of whose cubes is 152.

11. Find three consecutive numbers such that twice the product of the first and third is equal to the square of the second increased by 62.

12. A grazier bought a certain number of oxen for \$240. Having lost 3, he sold the remainder at \$8 a head more than they cost him, and gained \$59. How many did he buy?

13. A merchant bought a quantity of flour for \$96. If he had bought 8 barrels more for the same money, he would have paid \$2 less per barrel. How many barrels did he buy, and at what price?

14. Find two numbers, whose product is 78, such that if one be divided by the other the quotient is 2, and the remainder 1.

15. The plate of a rectangular looking-glass is 18 inches by 12. It is to be framed with a frame all parts of which are of the same width, and whose area is equal to that of the glass. Required the width of the frame.

16. A merchant sold a quantity of flour for \$39, and gained as much per cent as the flour cost him. What was the cost of the flour?

17. A certain company agreed to build a vessel for \$6300; but, two of their number having died, the rest had each to advance \$200 more than they otherwise would have done. Of how many persons did the company consist at first?

18. Divide the number 24 into two parts, such that the sum of the fractions obtained by dividing 24 by them shall be $\frac{64}{15}$.

19. A detachment from an army was marching in regular column, with 6 men more in depth than in front. When the enemy came in sight, the front was increased by 870 men, and the whole was thus drawn up in 4 lines. Required the number of men.

20. A merchant sold goods for \$16, and lost as much per cent as the goods cost him. What was the cost of the goods?

21. A certain farm is a rectangle, whose length is twice its breadth. If it should be enlarged 20 rods in length, and 24 rods in breadth, its area would be doubled. Of how many acres does the farm consist?

22. A square court-yard has a gravel-walk around it. The side of the court lacks one yard of being 6 times the breadth of the walk, and the number of square yards in the walk exceeds the number of yards in the perimeter of the court by 340. Find the area of the court and the width of the walk.

23. A merchant bought 54 bushels of wheat, and a certain quantity of barley. For the former he gave half as many dimes per bushel as there were bushels of barley, and for the latter 40 cents a bushel less. He sold the mixture at \$1 per bushel, and lost \$57.60 by the operation. Required the quantity of barley, and its price per bushel.

24. A certain number consists of two digits, the left-hand digit being twice the right-hand. If the digits are inverted, the product of the number thus formed, increased by 11, and the original number, is 4956. Find the number.

25. A cistern can be filled by two pipes running together in 2 hours 55 minutes. The larger pipe by itself will fill it sooner than the smaller by 2 hours. What time will each pipe separately take to fill it?

26. A and B gained by trade \$1800. A's money was in the firm 12 months, and he received for his principal and gain \$2600. B's money, which was \$3000, was in the firm 16 months. How much money did A put into the firm?

27. My gross income is \$1000. After deducting a percentage for income tax, and then a percentage, less by one than that of the income tax, from the remainder, the income is reduced to \$912. Find the rate per cent of the income tax.

28. A man travelled 102 miles. If he had gone 3 miles more an hour, he would have performed the journey in $5\frac{2}{3}$ hours less time. How many miles an hour did he go?

29. The number of square inches in the surface of a cubical block exceeds the number of inches in the sum of its edges by 210. What is its volume?

30. A man has two square lots of unequal size, containing together 15,025 square feet. If the lots were contiguous, it would require 530 feet of fence to embrace them in a single enclosure of six sides. Required the area of each lot.

31. A set out from C towards D at the rate of 3 miles an hour. After he had gone 28 miles, B set out from D towards C, and went every hour $\frac{1}{19}$ of the entire distance; and after he had travelled as many hours as he went miles in an hour, he met A. Required the distance from C to D.

32. A courier proceeds from P to Q in 14 hours. A second courier starts at the same time from a place 10 miles behind P, and arrives at Q at the same time as the first courier. The second courier finds that he takes half an hour less than the first to accomplish 20 miles. Find the distance from P to Q.

33. A person bought a number of $20 mining-shares when they were at a certain rate per cent discount for $1500; and afterwards, when they were at the same rate per cent premium, sold them all but 60 for $1000. How many did he buy, and what did he give for each of them?

XXIII. EQUATIONS IN THE QUADRATIC FORM.

269. An equation is in the *quadratic form* when it is expressed in three terms, two of which contain the unknown quantity; and of these two, *one has an exponent twice as great as the other;* as,

$$x^6 - 6x^3 = 16;$$

$$x^3 + x^{\frac{3}{2}} = 72;$$

$$(x^2 - 1)^2 + 3(x^2 - 1) = 18; \text{ etc.}$$

270. The rules for the solution of quadratics are applicable to equations having the same form.

1. Solve the equation $x^6 - 6x^3 = 16$.

Completing the square,

$$x^6 - 6x^3 + 9 = 16 + 9 = 25.$$

Extracting the square root,

$$x^3 - 3 = \pm 5.$$

Whence, $x^3 = 3 \pm 5 = 8$ or -2.

Extracting the cube root, $x = 2$ or $-\sqrt[3]{2}$, *Ans.*

Note. There are also four imaginary roots, which may be obtained by the method explained in Art. 282.

2. Solve the equation $2x + 3\sqrt{x} = 27$.

Since $\sqrt{x}$ is the same as $x^{\frac{1}{2}}$, this is in the quadratic form.

Multiplying by 8, and adding 3^2 or 9 to both members,

$$16x + 24\sqrt{x} + 9 = 216 + 9 = 225.$$

Extracting the square root,

$$4\sqrt{x} + 3 = \pm 15.$$

Or, $4\sqrt{x} = -3 \pm 15 = 12$ or -18.

Whence, $\sqrt{x} = 3$ or $-\frac{9}{2}$.

Squaring, $x = 9$ or $\frac{81}{4}$, *Ans.*

3. Solve the equation $16x^{-\frac{3}{2}} - 22x^{-\frac{3}{4}} = 3$.

Multiplying by 16, and adding 11^2 to both members,

$$16^2x^{-\frac{3}{2}} - 16 \times 22x^{-\frac{3}{4}} + 121 = 48 + 121 = 169.$$

Extracting the square root,

$$16x^{-\frac{3}{4}} - 11 = \pm 13.$$

Or, $16x^{-\frac{3}{4}} = 11 \pm 13 = -2$ or 24.

Whence, $x^{-\frac{3}{4}} = -\frac{1}{8}$ or $\frac{3}{2}$.

Extracting the cube root,

$$x^{-\frac{1}{4}} = -\frac{1}{2} \text{ or } \left(\frac{3}{2}\right)^{\frac{1}{3}}.$$

Raising to the fourth power,

$$x^{-1} = \frac{1}{16} \text{ or } \left(\frac{3}{2}\right)^{\frac{4}{3}}.$$

Inverting both members, $x = 16$ or $\left(\frac{2}{3}\right)^{\frac{4}{3}}$, *Ans.*

Note. In solving equations of the form $x^{\frac{p}{q}} = a$, first extract the root corresponding to the numerator, and afterwards raise to the power corresponding to the denominator. Particular attention should be paid to the algebraic signs; see Arts 192 and 201.

EXAMPLES.

Solve the following equations:

4. $x^4 - 25x^2 = -144$.

5. $x^6 + 20x^3 - 69 = 0$.

6. $x^{10} + 31x^5 - 32 = 0$.

7. $x^{-4} - 9x^{-2} = -20$.

8. $81x^2 + \frac{1}{x^2} = 82$.

9. $8x^6 - 216 = 37x^3$.

10. $(3x^2-2)^2-11(3x^2-2)+10=0.$

11. $(x^3-5)^2=241-29x^3.$

12. $x^3-x^{\frac{3}{2}}=56.$

13. $x^{\frac{6}{5}}+x^{\frac{3}{5}}=756.$

14. $2x^{\frac{2}{n}}+3x^{\frac{4}{n}}-56=0.$

15. $3x^{\frac{5}{3}}+x^{\frac{5}{6}}=3104.$

16. $3x^{\frac{3}{2}}+26x^{\frac{3}{4}}=-16.$

17. $2x^{-5}+61x^{-\frac{5}{2}}-96=0.$

18. $4x-15=17\sqrt{x}.$

19. $\dfrac{\sqrt{4x}+2}{4+\sqrt{x}}=\dfrac{4}{\sqrt{x}}-1.$

20. $3x^{\frac{4}{3}}-\dfrac{5x^{\frac{8}{3}}}{2}=-592.$

21. $8x^{-\frac{6}{5}}-15x^{-\frac{3}{5}}-2=0.$

271. An equation may be solved with reference to an expression, by regarding it as a single quantity.

1. Solve the equation $(x-5)^3-3(x-5)^{\frac{3}{2}}=40.$

Regarding $x-5$ as a single quantity, we complete the square in the usual way. Multiplying by 4, and adding 9 to both members,

$$4(x-5)^3-12(x-5)^{\frac{3}{2}}+9=160+9=169.$$

Extracting the square root,

$$2(x-5)^{\frac{3}{2}}-3=\pm 13.$$

Or, $$2(x-5)^{\frac{3}{2}}=3\pm 13=16 \text{ or } -10.$$

Whence, $$(x-5)^{\frac{3}{2}}=8 \text{ or } -5.$$

Extracting the cube root,

$$(x-5)^{\frac{1}{2}}=2 \text{ or } -\sqrt[3]{5}.$$

Squaring, $$x-5=4 \text{ or } \sqrt[3]{25}.$$

Transposing, $$x=9 \text{ or } 5+\sqrt[3]{25},\ \textit{Ans.}$$

An equation of the fourth degree may sometimes be solved by expressing it in the quadratic form.

2. Solve the equation $x^4 + 12x^3 + 34x^2 - 12x - 35 = 0$.

We may write the equation as follows:

$$(x^4 + 12x^3 + 36x^2) - 2x^2 - 12x = 35.$$

Or, $$(x^2 + 6x)^2 - 2(x^2 + 6x) = 35.$$

Completing the square,

$$(x^2 + 6x)^2 - 2(x^2 + 6x) + 1 = 36.$$

Extracting the square root, $(x^2 + 6x) - 1 = \pm 6$.

Whence, $x^2 + 6x = 1 \pm 6 = 7$ or -5.

Completing the square, $x^2 + 6x + 9 = 16$ or 4.

Extracting the square root,

$$x + 3 = \pm 4 \text{ or } \pm 2.$$

Whence, $$x = -3 \pm 4 \text{ or } -3 \pm 2$$

$$= 1,\ -7,\ -1, \text{ or } -5,\ \textit{Ans.}$$

Note. In solving equations like the above, the first step is to form a perfect square with the x^4 and x^3 terms, and a portion of the x^2 term. By Art. 261, the third term of the square is the square of the quotient obtained by dividing the x^3 term by twice the square root of the x^4 term.

3. Solve the equation $2x^2 + \sqrt{2x^2 + 1} = 11$.

Adding 1 to both members,

$$(2x^2 + 1) + \sqrt{2x^2 + 1} = 12.$$

Completing the square,

$$4(2x^2 + 1) + 4\sqrt{2x^2 + 1} + 1 = 48 + 1 = 49.$$

Extracting the square root,

$$2\sqrt{2x^2 + 1} + 1 = \pm 7.$$

Or, $$2\sqrt{2x^2 + 1} = -1 \pm 7 = 6 \text{ or } -8.$$

Whence, $$\sqrt{2x^2 + 1} = 3 \text{ or } -4.$$

Squaring, $$2x^2+1=9 \text{ or } 16.$$
$$2x^2=8 \text{ or } 15.$$
$$x^2=4 \text{ or } \frac{15}{2}.$$

Extracting the square root, $x=\pm 2$ or $\pm\frac{1}{2}\sqrt{30}$, *Ans.*

Note. In solving equations of this form, add such quantities to both members that the expression without the radical in the first member may be the same as that within, or some multiple of it.

EXAMPLES.

Solve the following equations:

4. $(x^2-5x)^2-8(x^2-5x)=84.$

5. $x^4+10x^3+17x^2-40x-84=0.$

6. $x^2-10x-2\sqrt{x^2-10x+18}+15=0.$

7. $x^2+5+\sqrt{x^2+5}=12.$

8. $2x^2+3x-5\sqrt{2x^2+3x+9}=-3.$

9. $x^4+2x^3-25x^2-26x+120=0.$

10. $x^4-6x^3-29x^2+114x=80.$

11. $x^2-6x+5\sqrt{x^2-6x+20}=46.$

12. $\sqrt{x+10}-\sqrt[4]{x+10}=2.$

13. $4x^2+6\sqrt{4x^2+12x-2}=-3-12x.$

14. $(x^3+16)^{\frac{2}{3}}-3(x^3+16)^{\frac{1}{3}}+2=0.$

15. $4(x-1)^{\frac{4}{3}}-5(x-1)^{\frac{2}{3}}+1=0.$

16. $x^4+14x^3+47x^2-14x-48=0.$

17. $3(x^2+5x)-2\sqrt{x^2+5x+1}=2.$

18. $(x-a)^{\frac{5}{3}}+2\sqrt{b(x-a)^{\frac{5}{6}}}-3b=0.$

XXIV. SIMULTANEOUS EQUATIONS.

INVOLVING QUADRATICS.

272. The degree of an equation containing more than one unknown quantity is determined by the greatest sum of the exponents of the unknown quantities in any term. Thus,

$2x + 3xy = 4$ is an equation of the second degree.

$x^2 - ay^2z = ab^3$ is an equation of the third degree.

Note. This definition assumes that the equation has been cleared of fractions, and freed from radical signs and fractional and negative exponents.

273. Two equations of the second degree with two unknown quantities will generally produce, by elimination, an equation of the fourth degree with one unknown quantity. The rules for quadratics are, therefore, not sufficient to solve all simultaneous equations of the second degree.

In several cases, however, the solution may be effected by the ordinary rules.

Case I.

274. *When one equation is of the first degree.*

Equations of this kind may always be solved by finding the value of one of the unknown quantities in terms of the other from the simple equation, and substituting the result in the other equation.

1. Solve the equations $\begin{cases} 2x^2 - xy = 6y. & (1) \\ x + 2y = 7. & (2) \end{cases}$

From (2), $2y = 7 - x$, or $y = \dfrac{7-x}{2}$. (3)

Substituting in (1), $2x^2 - x\left(\dfrac{7-x}{2}\right) = 6\left(\dfrac{7-x}{2}\right)$.

Clearing of fractions, $4x^2 - 7x + x^2 = 42 - 6x$.

Or, $$5x^2 - x = 42.$$

Solving this equation, $$x = 3 \text{ or } -\frac{14}{5}.$$

Substituting in (3), $$y = \frac{7-3}{2} \text{ or } \frac{7+\frac{14}{5}}{2}$$

$$= 2 \text{ or } \frac{49}{10}.$$

Ans. $x = 3,\ y = 2$; or, $x = -\frac{14}{5},\ y = \frac{49}{10}$.

EXAMPLES.

Solve the following equations:

2. $\begin{cases} 2x^2 - 3y^2 = -10. \\ 3x + y = 1. \end{cases}$

3. $\begin{cases} x + y = -1. \\ xy = -56. \end{cases}$

4. $\begin{cases} x - y = 3. \\ x^2 + y^2 = 117. \end{cases}$

5. $\begin{cases} 10x + y = 3xy. \\ x - y = -2. \end{cases}$

6. $\begin{cases} x^3 - y^3 = -37. \\ x - y = -1. \end{cases}$

7. $\begin{cases} x - y = 5. \\ xy = -6. \end{cases}$

8. $\begin{cases} x + y = 3. \\ x^2 + y^2 = 29. \end{cases}$

9. $\begin{cases} \frac{x}{2} + \frac{y}{3} = 4. \\ \frac{2}{x} + \frac{3}{y} = 1. \end{cases}$

10. $\begin{cases} x^3 + y^3 = 152. \\ x + y = 2. \end{cases}$

11. $\begin{cases} 3x^2 - 2xy = 15. \\ 2x + 3y = 12. \end{cases}$

12. $\begin{cases} 8x^3 - y^3 = -7. \\ 2x - y = -1. \end{cases}$

13. $\begin{cases} x^2 + 3xy - y^2 = 23. \\ x + 2y = 7. \end{cases}$

14. $\begin{cases} \frac{x}{y} + \frac{y}{x} = \frac{5}{2}. \\ 3x - 2y = -4. \end{cases}$

Case II.

275. *When the equations are symmetrical with respect to x and y.*

Note 1. An equation is symmetrical with respect to two quantities when they can be interchanged without destroying the equality.

Thus, $x^2 - xy + y^2 = 3$ is symmetrical, for on interchanging x and y it becomes $y^2 - yx + x^2 = 3$, which is equivalent to the first equation. But $x - y = 1$ is not symmetrical, for on interchanging x and y it becomes $y - x = 1$, which is a different equation.

In solving equations by the symmetrical method, they must be combined in such a way as to give the values of the sum and difference of the unknown quantities.

1. Solve the equations
$$\begin{cases} x + y = 2. & (1) \\ xy = -15. & (2) \end{cases}$$

Squaring (1), $$x^2 + 2xy + y^2 = 4. \qquad (3)$$

Multiplying (2) by 4, $$4xy = -60. \qquad (4)$$

Subtracting (4) from (3), $$x^2 - 2xy + y^2 = 64.$$

Extracting the square root, $$x - y = \pm 8. \qquad (5)$$

Adding (1) and (5), $$2x = 2 \pm 8 = 10 \text{ or } -6.$$

Whence, $$x = 5 \text{ or } -3.$$

Subtracting (5) from (1), $$2y = 2 \mp 8 = -6 \text{ or } 10.$$

Whence, $$y = -3 \text{ or } 5.$$

Ans. $x = 5$, $y = -3$; or, $x = -3$, $y = 5$.

Note 2. The signs $\pm$ and $\mp$ before two quantities signify that when the first quantity is $+$, the second is $-$; and when the first is $-$, the second is $+$. Thus, in the above solution, when $2x = 2 + 8$, $2y = 2 - 8$; and when $2x = 2 - 8$, $2y = 2 + 8$. That is, when $x = 5$, $y = -3$; and when $x = -3$, $y = 5$.

In the operation, the sign $\pm$ is changed to $\mp$ whenever $+$ would be changed to $-$.

Note 3. The above equations may also be solved as in Case I.; but the symmetrical method is shorter, and more elegant.

2. Solve the equations $\begin{cases} x^2 + y^2 = 50. & (1) \\ x - y = 8. & (2) \end{cases}$

Squaring (2), $x^2 - 2xy + y^2 = 64.$ (3)

Subtracting (3) from (1), $2xy = -14.$ (4)

Adding (1) and (4), $x^2 + 2xy + y^2 = 36.$

Whence, $x + y = \pm 6.$ (5)

Adding (2) and (5), $2x = 8 \pm 6 = 14$ or $2.$

Whence, $x = 7$ or $1.$

Subtracting (2) from (5), $2y = -8 \pm 6 = -2$ or $-14.$

Whence, $y = -1$ or $-7.$

Ans. $x = 7,\ y = -1$; or, $x = 1,\ y = -7.$

Note 4. The symmetrical method may often be used in cases like the above, where the equations are symmetrical except in the signs of the terms.

3. Solve the equations $\begin{cases} x^3 + y^3 = 133. & (1) \\ x^2 - xy + y^2 = 19. & (2) \end{cases}$

Dividing (1) by (2), $x + y = 7.$ (3)

Squaring, $x^2 + 2xy + y^2 = 49.$ (4)

Subtracting (4) from (2), $-3xy = -30.$

Or, $-xy = -10.$ (5)

Adding (2) and (5), $x^2 - 2xy + y^2 = 9.$

Whence, $x - y = \pm 3.$ (6)

Adding (3) and (6), $2x = 7 \pm 3 = 10$ or $4.$

Whence, $x = 5$ or $2.$

Subtracting (6) from (3), $2y = 7 \mp 3 = 4$ or $10.$

Whence, $y = 2$ or $5.$

Ans. $x = 5,\ y = 2$; or, $x = 2,\ y = 5.$

EXAMPLES.

Solve the following equations:

4. $\begin{cases} x+y=1. \\ xy=-6. \end{cases}$

5. $\begin{cases} x-y=6. \\ x^2+y^2=90. \end{cases}$

6. $\begin{cases} x-y=-10. \\ xy=-21. \end{cases}$

7. $\begin{cases} x^3+y^3=-19. \\ x^2-xy+y^2=19. \end{cases}$

8. $\begin{cases} x^2+y^2=25. \\ xy=12. \end{cases}$

9. $\begin{cases} x+y=-4. \\ x^2+y^2=58. \end{cases}$

10. $\begin{cases} x^3-y^3=98. \\ x-y=2. \end{cases}$

11. $\begin{cases} x^3+y^3=9. \\ x+y=3. \end{cases}$

12. $\begin{cases} x-y=\frac{1}{2}. \\ xy=60. \end{cases}$

13. $\begin{cases} x^2+y^2=85. \\ xy=42. \end{cases}$

14. $\begin{cases} x^3-y^3=-316. \\ x-y=-4. \end{cases}$

15. $\begin{cases} x^2+y^2=193. \\ x+y=-5. \end{cases}$

16. $\begin{cases} x+y=12. \\ xy=-45. \end{cases}$

17. $\begin{cases} x^3-y^3=-65. \\ x^2+xy+y^2=13. \end{cases}$

18. $\begin{cases} x^2+y^2=157. \\ x-y=-5. \end{cases}$

19. $\begin{cases} x^3+y^3=-386. \\ x+y=-2. \end{cases}$

CASE III.

276. *When each equation is of the second degree, and homogeneous* (Art. 35).

Note. Certain examples, in which the equations are of the second degree and homogeneous, may be solved by the method of Case II. The method of Case III. should be used only when the equations can be solved in no other way.

1. Solve the equations $\begin{cases} x^2 - 2xy = 5. \\ x^2 + y^2 = 29. \end{cases}$

Putting $y = vx$ in the given equations, we have

$$x^2 - 2vx^2 = 5; \text{ or, } x^2 = \frac{5}{1-2v}. \qquad (1)$$

$$x^2 + v^2x^2 = 29; \text{ or, } x^2 = \frac{29}{1+v^2}.$$

Equating the values of x^2, $\frac{5}{1-2v} = \frac{29}{1+v^2}$.

$$5 + 5v^2 = 29 - 58v.$$

$$5v^2 + 58v = 24.$$

Solving this equation, $v = \frac{2}{5}$ or -12.

Substituting these values in (1), $x^2 = \dfrac{5}{1-\frac{4}{5}}$ or $\dfrac{5}{1+24}$

$$= 25 \text{ or } \frac{1}{5}.$$

Whence, $x = \pm 5$ or $\pm \frac{1}{\sqrt{5}}$.

Substituting the values of v and x in the equation $y = vx$,

$$y = \frac{2}{5}(\pm 5) \text{ or } -12\left(\pm \frac{1}{\sqrt{5}}\right) = \pm 2 \text{ or } \mp \frac{12}{\sqrt{5}}.$$

Ans. $x = \pm 5, \quad y = \pm 2;$

or, $x = \pm \frac{1}{5}\sqrt{5}$, $y = \mp \frac{12}{5}\sqrt{5}$.

Note. In finding y from the equation $y = vx$, care must be taken to multiply each pair of values of x by the *corresponding value* of v.

EXAMPLES.

Solve the following equations:

2. $\begin{cases} x^2 - xy = 35. \\ xy + y^2 = 18. \end{cases}$

3. $\begin{cases} 2x^2 + xy = 15. \\ x^2 - y^2 = 8. \end{cases}$

4. $\begin{cases} x^2 + xy - y^2 = -11. \\ x^2 + y^2 = 13. \end{cases}$

5. $\begin{cases} x^2 + xy + 4y^2 = 6. \\ 3x^2 + 8y^2 = 14. \end{cases}$

6. $\begin{cases} x^2 + xy = 12. \\ xy - y^2 = 2. \end{cases}$

7. $\begin{cases} 2y^2 - 4xy + 3x^2 = 17. \\ y^2 - x^2 = 16. \end{cases}$

8. $\begin{cases} 2x^2 - 2xy - y^2 = 3. \\ x^2 + 3xy + y^2 = 11. \end{cases}$

9. $\begin{cases} 6x^2 - 5xy + 2y^2 = 12. \\ 3x^2 + 2xy - 3y^2 = -3. \end{cases}$

10. $\begin{cases} x^2 + xy - y^2 = 1. \\ x^2 - xy + 2y^2 = 8. \end{cases}$

11. $\begin{cases} 4xy - x^2 = 5. \\ 13x^2 - 31xy + 16y^2 = 2\frac{1}{2}. \end{cases}$

MISCELLANEOUS EXAMPLES.

277. No general rules can be given for the solution of examples which do not come under the cases just considered. Various artifices are employed, familiarity with which can only be obtained by experience.

1. Solve the equations $\begin{cases} x^3 - y^3 = 19. & (1) \\ x^2y - xy^2 = 6. & (2) \end{cases}$

Multiplying (2) by 3, $3x^2y - 3xy^2 = 18.$ (3)

Subtracting (3) from (1),

$$x^3 - 3x^2y + 3xy^2 - y^3 = 1.$$

Extracting the cube root, $x - y = 1.$ (4)

Dividing (2) by (4), $xy = 6.$ (5)

Equations (4) and (5) may now be solved by the method of Case II. We shall find $x = 3$ or -2, and $y = 2$ or -3.

Ans. $x = 3,\ y = 2$; or, $x = -2,\ y = -3$.

2. Solve the equations $\begin{cases} x^3 + y^3 = 9xy. \\ x + y = 6. \end{cases}$

Putting $x = u + v$ and $y = u - v$, we have

$$(u + v)^3 + (u - v)^3 = 9(u+v)(u-v). \quad (1)$$

$$(u + v) + (u - v) = 6. \quad (2)$$

Reducing (2), $\qquad 2u = 6$, or $u = 3$.

Reducing (1), $\qquad 2u^3 + 6uv^2 = 9(u^2 - v^2)$.

Substituting the value of u,

$$54 + 18v^2 = 9(9 - v^2).$$

Whence, $\qquad v^2 = 1$, or $v = \pm 1$.

Therefore, $\qquad x = u + v = 3 \pm 1 = 4$ or 2,

$$y = u - v = 3 \mp 1 = 2 \text{ or } 4.$$

Ans. $x = 4$, $y = 2$; or, $x = 2$, $y = 4$.

Note. The artifice of substituting $u + v$ and $u - v$ for x and y is advantageous in any case where the given equations are *symmetrical.*

3. Solve the equations

$$\begin{cases} x^2 + y^2 + 2x + 2y = 23. & (1) \\ xy = 6. & (2) \end{cases}$$

Multiplying (2) by 2, $\qquad 2xy = 12. \quad (3)$

Adding (1) and (3),

$$x^2 + 2xy + y^2 + 2x + 2y = 35.$$

Or, $\qquad (x + y)^2 + 2(x + y) = 35.$

Completing the square,

$$(x + y)^2 + 2(x + y) + 1 = 36.$$

Whence, $\qquad (x + y) + 1 = \pm 6,$

$$x + y = -1 \pm 6 = 5 \text{ or } -7. \quad (4)$$

Squaring (4), $\qquad x^2 + 2xy + y^2 = 25$ or $49. \quad (5)$

Multiplying (2) by 4, $\qquad 4xy = 24. \quad (6)$

Subtracting (6) from (5),

$$x^2 - 2xy + y^2 = 1 \text{ or } 25.$$

Whence, $x-y=\pm 1$ or ± 5. (7)

Adding (4) and (7), $2x=5\pm 1$ or -7 ± 5,

$x=3,\ 2,\ -1$, or -6.

Subtracting (7) from (4), $2y=5\mp 1$, or -7 ∓ 5,

$y=2,\ 3,\ -6$, or -1.

Ans. $x=3,\ y=2$; $x=2,\ y=3$;

$x=-1,\ y=-6$; or, $x=-6,\ y=-1$.

4. Solve the equations $\begin{cases} x^4+y^4=97. \\ x+y=-1. \end{cases}$

Putting $x=u+v$ and $y=u-v$, we have,

$$(u+v)^4+(u-v)^4=97. \qquad (1)$$

$$(u+v)+(u-v)=-1. \qquad (2)$$

Reducing (2), $2u=-1$, or $u=-\frac{1}{2}$.

Reducing (1), $2u^4+12u^2v^2+2v^4=97$.

Substituting the value of u,

$$\frac{1}{8}+3v^2+2v^4=97.$$

Solving this equation, $v^2=\frac{25}{4}$, or $-\frac{31}{4}$.

Whence, $v=\pm\frac{5}{2}$, or $\pm\frac{\sqrt{-31}}{2}$.

Therefore, $x=u+v=-\frac{1}{2}\pm\frac{5}{2}$ or $-\frac{1}{2}\pm\frac{\sqrt{-31}}{2}$

$$=2 \text{ or } -3 \text{ or } \frac{-1\pm\sqrt{-31}}{2};$$

and $y=u-v=-\frac{1}{2}\mp\frac{5}{2}$ or $-\frac{1}{2}\mp\frac{\sqrt{-31}}{2}$

$$=-3 \text{ or } 2 \text{ or } \frac{-1\mp\sqrt{-31}}{2}.$$

EXAMPLES.

Solve the following equations:

5. $\begin{cases} xy - 2x = 5. \\ xy + 3y = -2. \end{cases}$

6. $\begin{cases} x + y = 9. \\ \sqrt[3]{x} + \sqrt[3]{y} = 3. \end{cases}$

7. $\begin{cases} 4x^2 - 3y^2 = -11. \\ 11x^2 + 5y^2 = 301. \end{cases}$

8. $\begin{cases} x^3 + y^3 = 35. \\ x^2y + xy^2 = 30. \end{cases}$

9. $\begin{cases} \dfrac{x}{y} + \dfrac{y}{x} = \dfrac{29}{10}. \\ 3x - 2y = 4. \end{cases}$

10. $\begin{cases} x^2 + y^2 = 5m^2. \\ x - y = m. \end{cases}$

11. $\begin{cases} x^2 + y^2 + x + y = 18. \\ xy = 6. \end{cases}$

12. $\begin{cases} x^2 - 2xy = 16. \\ 2xy + y^2 = -3. \end{cases}$

13. $\begin{cases} x^3 + y^3 = 18xy. \\ x + y = 12. \end{cases}$

14. $\begin{cases} x^2 + 3xy = -14. \\ xy + 4y^2 = 30. \end{cases}$

15. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 11. \\ \dfrac{1}{xy} = 18. \end{cases}$

16. $\begin{cases} x - y = a - b. \\ xy = 2a^2 + 2ab. \end{cases}$

17. $\begin{cases} x^2 + y^2 = 9 - x. \\ x^2 - y^2 = 6. \end{cases}$

18. $\begin{cases} \dfrac{1}{x^2} + \dfrac{1}{y^2} = 65. \\ \dfrac{1}{x} - \dfrac{1}{y} = 11. \end{cases}$

19. $\begin{cases} x^2 + y^2 - x - y = 18. \\ xy + x + y = 19. \end{cases}$

20.* $\begin{cases} x^2 + xy + y^2 = 7. \\ x^4 + x^2y^2 + y^4 = 133. \end{cases}$

21. $\begin{cases} x^2y + xy^2 = 6. \\ \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{2}{3}. \end{cases}$

22. $\begin{cases} x^4 + y^4 = 17. \\ x - y = 3. \end{cases}$

23. $\begin{cases} x^3 - y^3 = 7a^3. \\ x - y = a. \end{cases}$

* Divide the second equation by the first.

24. $\begin{cases} x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 19. \\ x^2 + xy + y^2 = 133. \end{cases}$

25. $\begin{cases} x + 2y = 3a + b. \\ xy + y^2 = 2a(a + b). \end{cases}$

26. $\begin{cases} x^2y + xy^2 = 30. \\ x^4y^2 + x^2y^4 = 468. \end{cases}$

27. $\begin{cases} x^3 - y^3 = 6a^2b + 2b^3. \\ xy(x - y) = 2a^2b - 2b^3. \end{cases}$

28. $\begin{cases} x^2 + 3x + y = 73 - 2xy. \\ y^2 + 3y + x = 44. \end{cases}$

29. $\begin{cases} \dfrac{x}{y} + \dfrac{4\sqrt{x}}{\sqrt{y}} = \dfrac{33}{4}. \\ x - y = 5. \end{cases}$

30. $\begin{cases} x^2 - xy + y^2 = 19. \\ 2x^2 - y^2 = -17. \end{cases}$

31. $\begin{cases} x + y = 3(a - b). \\ xy = 2a^2 - 5ab + 2b^2. \end{cases}$

32. $\begin{cases} x^5 + y^5 = 33. \\ x + y = 3. \end{cases}$

33. $\begin{cases} xy^2 + y = 1. \\ x^2y^4 + y^2 = 5. \end{cases}$

34. $\begin{cases} x + z = 7. \\ 2y - 3z = -5. \\ x^2 + y^2 - z^2 = 11. \end{cases}$

35. $\begin{cases} 2x^2 - 7xy - 2y^2 = 5. \\ 3xy - x^2 + 6y^2 = 44. \end{cases}$

36. $\begin{cases} \dfrac{x - y}{x + y} - \dfrac{x + y}{x - y} = \dfrac{3}{2}. \\ 2x - y = 7. \end{cases}$

37. $\begin{cases} x^2 + y^2 = 7 + xy. \\ x^3 + y^3 = 6xy - 1. \end{cases}$

PROBLEMS.

278. Note. In the following problems, as in those of Chap. XXII., only those answers are to be retained which satisfy the conditions of the problem.

1. The sum of the squares of two numbers is 106, and the difference of their squares is $\frac{7}{2}$ the square of their difference. Find the numbers.

2. What two numbers are those whose difference multiplied by the less produces 42, and by their sum, 133?

3. The sum of the areas of two square fields is 1300 square rods, and it requires 200 rods of fence to enclose both. What are the areas of the fields?

4. The difference of the squares of two numbers is 7, and the product of their squares is 144. Find the numbers.

5. If the length of a rectangular field were increased by 2 rods, and its breadth by 3 rods, its area would be 108 square rods; and if its length were diminished by 2 rods, and its breadth by 3 rods, its area would be 24 square rods. Find the length and breadth of the field.

6. The sum of the cubes of two numbers is 407, and the sum of their squares exceeds their product by 37. Find the numbers.

7. A man bought 6 ducks and 2 turkeys for \$15. He bought four more ducks for \$14 than turkeys for \$9. What was the price of each?

8. Find a number of two figures, such that if its digits are inverted, the sum of the number thus formed, and the original number, is 33, and their product is 252.

9. The sum of the terms of a fraction is 8. If 1 is added to each term, the product of the resulting fraction and the original fraction is $\frac{2}{5}$. Required the fraction.

10. A rectangular garden is surrounded by a walk 7 feet wide; the area of the garden is 15,000 square feet, and of the walk 3696 square feet. Find the length and breadth of the garden.

11. A rectangular field contains 160 square rods. If its length be increased by 4 rods, and its breadth by 3 rods, its area is increased by 100 square rods. Find the length and breadth of the field.

12. A man rows down stream 12 miles in 4 hours less time than it takes him to return. Should he row at twice his ordinary rate, his rate down stream would be 10 miles an hour. Find his rate in still water, and the rate of the stream.

13. A and B bought a farm of 104 acres, for which they paid $320 each. On dividing the land, A said to B, "If you will let me have my portion in the situation which I shall choose, you shall have so much more land than I, that mine shall cost $3 an acre more than yours." B accepted the proposal. How much did each have, and at what price per acre?

14. If the product of two numbers be added to their sum, the result is 47; and the sum of their squares exceeds their sum by 62. Find the numbers.

Note. Let the numbers be represented by $x+y$ and $x-y$.

15. The sum of two numbers is 7, and the sum of their fourth powers is 641. Required the numbers.

16. The fore-wheel of a carriage makes 15 more revolutions than the hind-wheel in going 180 yards; but if the circumference of each wheel were increased by 3 feet, the fore-wheel would make only 9 more revolutions than the hind-wheel in the same distance. Find the circumference of each wheel.

17. A man has $1300, which he divides into two portions, and loans at different rates of interest, so that the two portions produce equal returns. If the first portion had been loaned at the second rate, it would have produced $36; and if the second portion had been loaned at the first rate, it would have produced $49. Required the rates of interest.

18. Cloth, when wetted, shrinks $\frac{1}{8}$ in its length and $\frac{1}{16}$ in its width. If the surface of a piece of cloth is diminished by $5\frac{3}{4}$ square yards, and the length of the four sides by $4\frac{1}{4}$ yards, what were the length and width of the cloth originally?

XXV. THEORY OF QUADRATIC EQUATIONS.

279. Denoting the roots of the equation $x^2+px=q$ by r_1 and r_2, we have (Art. 267),

$$r_1=\frac{-p+\sqrt{p^2+4q}}{2}, \text{ and } r_2=\frac{-p-\sqrt{p^2+4q}}{2}.$$

Adding these values,

$$r_1+r_2=\frac{-2p}{2}=-p.$$

Multiplying them together,

$$r_1r_2=\frac{p^2-(p^2+4q)}{4} \text{ (Art. 95) } =\frac{-4q}{4}=-q.$$

That is, *if a quadratic equation be reduced to the form* $x^2+px=q$, *the algebraic sum of the roots is equal to the coefficient of* x *with its sign changed, and the product of the roots is equal to the second member, with its sign changed.*

Example. Required the sum and product of the roots of the equation $2x^2-7x-15=0$.

The equation may be written in the form

$$x^2-\frac{7x}{2}=\frac{15}{2}.$$

Whence, the sum of the roots is $\frac{7}{2}$, and their product is $-\frac{15}{2}$.

280. The principles of Art. 279 may be used to form a quadratic equation which shall have any required roots.

For, denoting the roots of the equation $x^2+px-q=0$ by r_1 and r_2, we have, by the preceding article,

$$p=-(r_1+r_2), \text{ and } -q=r_1r_2.$$

We may therefore write the equation in the form

$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0,$$

or,

$$x^2 - r_1 x - r_2 x + r_1 r_2 = 0.$$

That is (Art. 105), $(x - r_1)(x - r_2) = 0.$

Hence, to form an equation which shall have any required roots,

Subtract each of the roots from x, and place the product of the resulting expressions equal to zero.

Example. Form the equation whose roots are 4 and $-\frac{7}{4}$.

By the rule, $(x-4)\left(x+\frac{7}{4}\right)=0.$

Multiplying by 4, $(x-4)(4x+7)=0.$

Or, $4x^2 - 9x - 28 = 0,$ *Ans.*

EXAMPLES.

281. Find by inspection the sum and product of the roots of:

1. $x^2 + 5x + 2 = 0.$

2. $x^2 - 7x + 11 = 0.$

3. $x^2 + 6x - 1 = 0.$

4. $2x^2 - 3x - 2 = 0.$

5. $8x^2 - x + 4 = 0.$

6. $6x - 4x^2 + 3 = 0.$

7. $7 - 12x - 14x^2 = 0.$

8. $4x^2 - 4ax + a^2 - b^2 = 0.$

Form the equations whose roots are:

9. 4, 5.

10. 1, -3.

11. 3, $-\frac{3}{5}$.

12. 7, $-\frac{19}{3}$.

13. $\frac{2}{3}, \frac{3}{4}$.

14. $-\frac{8}{3}, \frac{4}{7}$.

15. $-\frac{5}{3}, -\frac{7}{2}$.

16. $-\frac{17}{3}$, 0.

17. $a-b,\ a+2b$.

18. $m(1+m),\ m(1-m)$.

19. $2+\sqrt{3},\ 2-\sqrt{3}$.

20. $\frac{m+\sqrt{n}}{2}, \frac{m-\sqrt{n}}{2}$.

282. By Art. 280, the equation $x^2+px-q=0$ may be written in the form $(x-r_1)(x-r_2)=0$, where r_1 and r_2 are its roots.

It will be observed that the roots may be obtained by *placing the factors of the first member separately equal to zero, and solving the simple equations thus formed.*

This principle is often used in solving equations:

1. Solve the equation $(2x-3)(3x+5)=0$.

Placing the factors separately equal to zero,

$$2x-3=0, \text{ or } x=\frac{3}{2};$$

and $$3x+5=0, \text{ or } x=-\frac{5}{3}.$$

$$\textit{Ans. } x=\frac{3}{2}, \text{ or } -\frac{5}{3}.$$

2. Solve the equation $x^3-5x^2-24x=0$.

Factoring the first member, $x(x-8)(x+3)=0$.

Therefore, $$x=0;$$

$$x-8=0, \text{ or } x=8;$$

and $$x+3=0, \text{ or } x=-3.$$

$$\textit{Ans. } x=0,\ 8, \text{ or } -3.$$

3. Solve the equation $x^3+4x^2-x-4=0$.

Factcring the first member (Art. 105),

$$(x+4)(x^2-1)=0.$$

Therefore, $$x+4=0, \text{ or } x=-4;$$

and $$x^2-1=0, \text{ or } x=\pm 1.$$

$$\textit{Ans. } x=-4 \text{ or } \pm 1.$$

4. Solve the equation $$x^3 - 1 = 0.$$

Factoring the first member,

$$(x-1)(x^2+x+1)=0.$$

Therefore, $$x-1=0, \text{ or } x=1;$$

and $$x^2+x+1=0. \qquad (1)$$

Solving (1) by the rules for quadratics, $x = \dfrac{-1 \pm \sqrt{-3}}{2}.$

Ans. $x = 1$ or $\dfrac{-1 \pm \sqrt{-3}}{2}.$

EXAMPLES.

Solve the following equations:

5. $\left(x-\frac{3}{5}\right)\left(x+\frac{7}{2}\right)=0.$

6. $2x^2 - x = 0.$

7. $(ax+b)(bx-a)=0.$

8. $(x^2-4)(x^2-9)=0.$

9. $(x-2)(x^2+9x+20)=0.$

10. $2x^3 - 18x = 0.$

11. $(3x+1)(4x^2-25)=0.$

12. $3x^3 + 12x^2 = 0.$

13. $(x^2-a^2)(x^2-ax-b)=0.$

14. $24x^3 - 2x^2 - 12x = 0.$

15. $x(2x+5)(3x-7)(4x+1)=0.$

16. $(x^2-5x+6)(x^2+7x+12)(x^2-3x-4)=0.$

17. $x^3 + 1 = 0.$

18. $x^6 - 1 = 0.$

19. $x^3 - x^2 - 9x + 9 = 0.$

20. $2x^3 + 3x^2 - 2x - 3 = 0.$

Note. The above examples are illustrations of the important principle that, the degree of an equation indicates the number of its roots; thus, an equation of the third degree has three roots; of the fourth degree, four roots; etc.

It should be observed that the roots are not necessarily *unequal;* thus, the equation $x^2 - 2x + 1 = 0$ may be written $(x-1)(x-1)=0$, and therefore the two roots are 1 and 1.

FACTORING.

283. A *Quadratic Expression* is a trinomial expression of the form ax^2+bx+c. The principles of the preceding articles serve to resolve any such expression into two simple factors.

The expression ax^2+bx+c may be written

$$a\left(x^2+\frac{bx}{a}+\frac{c}{a}\right).$$

But, by Art. 280, $x^2+\frac{bx}{a}+\frac{c}{a}=(x-r_1)(x-r_2)$,

where r_1 and r_2 are the roots of the equation,

$$x^2+\frac{bx}{a}+\frac{c}{a}=0, \text{ or } ax^2+bx+c=0;$$

which, it will be observed, may be formed by placing the given expression equal to zero. Hence,

$$ax^2+bx+c=a(x-r_1)(x-r_2).$$

1. Factor $6x^2+11x+3$.

Placing the expression equal to 0,

$$6x^2+11x+3=0.$$

Solving, as in Art. 267,

$$x=\frac{-11\pm\sqrt{121-72}}{12}=\frac{-11\pm 7}{12}=-\frac{1}{3} \text{ or } -\frac{3}{2}.$$

Then, $\quad a=6,\ r_1=-\frac{1}{3},\ r_2=-\frac{3}{2}.$

Hence,

$$6x^2+11x+3=6\left(x+\frac{1}{3}\right)\left(x+\frac{3}{2}\right)$$

$$=3\left(x+\frac{1}{3}\right)2\left(x+\frac{3}{2}\right)$$

$$=(3x+1)(2x+3), \textit{ Ans.}$$

2. Factor $4+13x-12x^2$.

Solving the equation $4+13x-12x^2=0$, we have

$$x=\frac{-13\pm\sqrt{169+192}}{-24}=\frac{-13\pm 19}{-24}=-\frac{1}{4}\text{ or }\frac{4}{3}.$$

Hence,
$$\begin{aligned}4+13x-12x^2&=-12\left(x+\frac{1}{4}\right)\left(x-\frac{4}{3}\right)\\&=4\left(x+\frac{1}{4}\right)\left(-3\right)\left(x-\frac{4}{3}\right)\\&=(1+4x)(4-3x),\ \textit{Ans.}\end{aligned}$$

Note. It must be remembered, in using the formula $a(x-r_1)(x-r_2)$, that a represents the coefficient of x^2 in the given expression; thus, in Ex. 2, we have $a=-12$.

EXAMPLES.

Factor the following:

3. $x^2+13x+40$.

4. $x^2-11x+18$.

5. $x^2-4x-60$.

6. $2x^2-7x-15$.

7. $4x^2-15x+9$.

8. $5x^2+36x+7$.

9. $4x^2+15x-4$.

10. $39-10x-x^2$.

11. $2+x-6x^2$.

12. x^2-4x+1.

13. $9x^2-6x-4$.

14. $8x^2-18x+9$.

15. $6-x-2x^2$.

16. $7x^2+50x+7$.

17. $6x^2-13ax-15a^2$.

18. $5+4x-12x^2$.

19. $9x^2-12x+1$.

20. $12x^2-7xy-10y^2$.

21. $8x^2+18xy-5y^2$.

22. $10x^2-23x+6$.

23. $20x^2+41mx+20m^2$.

24. $16x^2-34x+15$.

25. $1-8x-x^2$.

26. $15b^2+26bx-24x^2$.

27. $21x^2+58mnx+21m^2n^2$.

28. $25x^2-20x-2$.

284. Many expressions may be factored by the artifice of completing the square, in connection with Art. 111.

1. Factor $a^4 + a^2b^2 + b^4$.

By Art. 108, the expression will become a perfect square if the middle term is $2a^2b^2$. Hence,

$$\begin{aligned} a^4 + a^2b^2 + b^4 &= (a^4 + 2a^2b^2 + b^4) - a^2b^2 \\ &= (a^2 + b^2)^2 - a^2b^2 \\ &= (a^2 + b^2 + ab)(a^2 + b^2 - ab), \text{ (Art. 111)} \\ &= (a^2 + ab + b^2)(a^2 - ab + b^2), \textit{ Ans.} \end{aligned}$$

2. Factor $9x^4 - 39x^2 + 25$.

$$\begin{aligned} 9x^4 - 39x^2 + 25 &= (9x^4 - 30x^2 + 25) - 9x^2 \\ &= (3x^2 - 5)^2 - 9x^2 \\ &= (3x^2 - 5 + 3x)(3x^2 - 5 - 3x) \\ &= (3x^2 + 3x - 5)(3x^2 - 3x - 5), \textit{ Ans.} \end{aligned}$$

3. Factor $x^4 - x^2 + 1$.

$$\begin{aligned} x^4 - x^2 + 1 &= (x^4 + 2x^2 + 1) - 3x^2 \\ &= (x^2 + 1)^2 - (x\sqrt{3})^2 \\ &= (x^2 + x\sqrt{3} + 1)(x^2 - x\sqrt{3} + 1), \textit{ Ans.} \end{aligned}$$

EXAMPLES.

Factor the following:

4. $x^4 + x^2 + 1$.
5. $x^4 - 7x^2 + 1$.
6. $4a^4 - 8a^2b^2 + b^4$.
7. $m^4 - 14m^2n^2 + n^4$.
8. $1 - 13b^2 + 4b^4$.
9. $x^4 - 12x^2y^2 + 4y^4$.
10. $4a^4 + 8a^2 + 9$.
11. $4m^4 - 24m^2 + 25$.
12. $a^4 - 5a^2x^2 + x^4$.
13. $x^4 + 1$.
14. $4a^4 + 15a^2b^2 + 16b^4$.
15. $16x^4 - 49m^2x^2 + 9m^4$.
16. $9x^4 - 6x^2 + 4$.
17. $9a^4 + 14a^2m^2 + 25m^4$.
18. $4 - 32n^2 + 49n^4$.
19. $16x^4 - 49x^2y^2 + 25y^4$.

DISCUSSION OF THE GENERAL EQUATION.

285. The roots of the equation $x^2 + px = q$ are

$$r_1 = \frac{-p + \sqrt{p^2 + 4q}}{2} \text{ and } r_2 = \frac{-p - \sqrt{p^2 + 4q}}{2}.$$

We will now discuss these values for different values of p and q.

I. *Suppose q positive.*

Since p^2 is essentially positive (Art. 192), the quantity under the radical sign is positive and greater than p^2. Therefore the value of the radical is greater than p. Hence, r_1 is positive and r_2 is negative.

If p is positive, r_2 is numerically greater than r_1; that is, the negative root is numerically the greater.

If p is zero, the roots are numerically equal.

If p is negative, r_1 is numerically greater than r_2; that is, the positive root is numerically the greater.

II. *Suppose $q = 0$.*

The quantity under the radical sign is now equal to p^2, so that the value of the radical is p.

If p is positive, $r_1 = 0$, and r_2 is negative.

If p is negative, r_1 is positive, and $r_2 = 0$.

III. *Suppose q negative and $4q$ numerically $< p^2$.*

The quantity under the radical sign is now positive and less than p^2. Therefore the value of the radical is less than p.

If p is positive, both roots are negative.

If p is negative, both roots are positive.

IV. *Suppose q negative and $4q$ numerically $= p^2$.*

The quantity under the radical sign is now equal to zero. Therefore the roots are equal; being negative if p is positive, and positive if p is negative.

V. *Suppose q negative and $4q$ numerically $> p^2$.*

The quantity under the radical sign is now negative; hence, by Art. 201, both roots are imaginary.

The roots are both *rational* or both *irrational* according as $p^2 + 4q$ *is* or *is not* a perfect square.

EXAMPLES.

286. 1. Determine by inspection the nature of the roots of the equation $2x^2 - 5x - 18 = 0$.

The equation may be written $x^2 - \frac{5x}{2} = 9$.

Since q is positive and p negative, the roots are one positive and the other negative; and the positive root is numerically the greater.

In this case, $p^2 + 4q = \frac{25}{4} + 36 = \frac{169}{4}$; a perfect square. Hence the roots are both rational.

Determine by inspection the nature of the roots of the following:

2. $x^2 + 2x - 15 = 0$.

3. $x^2 + 5x + 6 = 0$.

4. $x^2 - 10x = -25$.

5. $3x^2 - 5x + 4 = 0$.

6. $6x^2 - 7x - 5 = 0$.

7. $9x^2 + 30x = -25$.

8. $9x^2 + 8 = 18x$.

9. $10 - 3x - 18x^2 = 0$.

XXVI. RATIO AND PROPORTION.

287. The **Ratio** of one quantity to another of the same kind is the quotient obtained by dividing the first quantity by the second.

Thus, the ratio of a to b is $\frac{a}{b}$; which is also expressed $a : b$.

288. The first term of a ratio is called the *antecedent*, and the second term the *consequent*.

Thus, in the ratio $a : b$, a is the antecedent and b the consequent.

289. A **Proportion** is an equality of ratios.

Thus, if the ratio of a to b is equal to the ratio of c to d, they form a proportion, which may be written in either of the forms:

$$a : b = c : d, \quad \frac{a}{b} = \frac{c}{d}, \quad \text{or} \quad a : b :: c : d.$$

290. The first and fourth terms of a proportion are called the *extremes*; and the second and third terms the *means*.

Thus, in the proportion $a : b = c : d$, a and d are the extremes, and b and c the means.

291. In a proportion in which the means are equal, either mean is called a **Mean Proportional** between the first and last terms, and the last term is called a **Third Proportional** to the first and second terms.

A **Fourth Proportional** to three quantities is the fourth term of a proportion whose first three terms are the three quantities taken in their order.

Thus, in the proportion $a : b = b : c$, b is a mean proportional between a and c, and c is a third proportional to a and b. In the proportion $a : b = c : d$, d is a fourth proportional to a, b, and c.

292. **A Continued Proportion** is one in which each consequent is the same as the next antecedent; as,

$$a:b=b:c=c:d=d:e.$$

PROPERTIES OF PROPORTIONS.

293. *In any proportion the product of the extremes is equal to the product of the means.*

Let the proportion be $a:b=c:d$.

Then, by Art. 289, $\frac{a}{b}=\frac{c}{d}$.

Clearing of fractions, $ad=bc$.

294. *A mean proportional between two quantities is equal to the square root of their product.*

Let the proportion be $a:b=b:c$.

Then, by Art. 293, $b^2=ac$.

Whence, $b=\sqrt{ac}$.

295. From the equation $ad=bc$, we obtain

$$a=\frac{bc}{d} \text{ and } b=\frac{ad}{c}.$$

That is, in any proportion either extreme is equal to the product of the means divided by the other extreme; and either mean is equal to the product of the extremes divided by the other mean.

296. (Converse of Art. 293.) *If the product of two quantities be equal to the product of two others, one pair may be made the extremes, and the other pair the means, of a proportion.*

Let $ad=bc$.

Dividing by bd, $\frac{ad}{bd}=\frac{bc}{bd}$, or $\frac{a}{b}=\frac{c}{d}$.

Whence, $a:b=c:d$.

In a similar manner we may prove that:

$$a : c = b : d,$$

$$b : d = a : c,$$

$$c : d = a : b, \text{ etc.}$$

297. *In any proportion the terms are in proportion by* **Alternation**; *that is, the first term is to the third, as the second term is to the fourth.*

Let $a : b = c : d.$

Then, by Art. 293, $ad = bc.$

Whence, by Art. 296, $a : c = b : d.$

298. *In any proportion the terms are in proportion by* **Inversion**; *that is, the second term is to the first, as the fourth term is to the third.*

Let $a : b = c : d.$

Then, $ad = bc.$

Whence, $b : a = d : c.$

299. *In any proportion the terms are in proportion by* **Composition**; *that is, the sum of the first two terms is to the first term, as the sum of the last two terms is to the third term.*

Let $a : b = c : d.$

Then, $ad = bc.$

Adding both members to ac,

$$ac + ad = ac + bc,$$

or, $$a(c + d) = c(a + b).$$

Whence (Art. 296),

$$a + b : a = c + d : c.$$

Similarly we may prove that

$$a + b : b = c + d : d.$$

300. *In any proportion the terms are in proportion by* **Division**; *that is, the difference of the first two terms is to the first term, as the difference of the last two terms is to the third term.*

Let $$a : b = c : d.$$

Then, $$ad = bc.$$

Subtracting both members from ac,

$$ac - ad = ac - bc,$$

or, $$a(c - d) = c(a - b).$$

Whence, $$a - b : a = c - d : c.$$

Similarly, $$a - b : b = c - d : d.$$

301. *In any proportion the terms are in proportion by* **Composition and Division**; *that is, the sum of the first two terms is to their difference, as the sum of the last two terms is to their difference.*

Let $$a : b = c : d.$$

Then, by Art. 299, $$\frac{a+b}{a} = \frac{c+d}{c}. \quad (1)$$

And, by Art. 300, $$\frac{a-b}{a} = \frac{c-d}{c}. \quad (2)$$

Dividing (1) by (2), $$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

Whence, $$a + b : a - b = c + d : c - d.$$

302. *In a series of equal ratios, any antecedent is to its consequent, as the sum of all the antecedents is to the sum of all the consequents.*

Let $$a : b = c : d = e : f.$$

Then, by Art. 293, $$ad = bc,$$

and $$af = be.$$

Also, $$ab = ba.$$

Adding, $$a(b + d + f) = b(a + c + e).$$

Whence (Art. 296), $$a : b = a + c + e : b + d + f.$$

303. *In any number of proportions, the products of the corresponding terms are in proportion.*

Let $a : b = c : d,$

and $e : f = g : h.$

Then, $\frac{a}{b} = \frac{c}{d}$, and $\frac{e}{f} = \frac{g}{h}.$

Multiplying these equals,

$$\frac{a}{b} \times \frac{e}{f} = \frac{c}{d} \times \frac{g}{h}, \text{ or } \frac{ae}{bf} = \frac{cg}{dh}.$$

Whence, $ae : bf = cg : dh.$

304. *In any proportion, like powers or like roots of the terms are in proportion.*

Let $a : b = c : d.$

Then, $\frac{a}{b} = \frac{c}{d}.$

Therefore, $\frac{a^n}{b^n} = \frac{c^n}{d^n}.$

Whence, $a^n : b^n = c^n : d^n.$

In a similar manner we may prove that

$$\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{c} : \sqrt[n]{d}.$$

305. *In any proportion, if the first two terms be multiplied by any quantity, as also the last two, the resulting quantities will be in proportion.*

Let $a : b = c : d.$

Then, $\frac{a}{b} = \frac{c}{d}.$

Therefore, $\frac{ma}{mb} = \frac{nc}{nd}.$

Whence, $ma : mb = nc : nd.$

In a similar manner we may prove that

$$\frac{a}{m} : \frac{b}{m} = \frac{c}{n} : \frac{d}{n}.$$

Note. Either m or n may be unity; that is, either couplet may be multiplied or divided without multiplying or dividing the other.

306. *In any proportion, if the first and third terms be multiplied by any quantity, as also the second and fourth terms, the resulting quantities will be in proportion.*

Let $$a : b = c : d.$$

Then, $$\frac{a}{b} = \frac{c}{d}.$$

Therefore, $$\frac{ma}{nb} = \frac{mc}{nd}.$$

Whence, $$ma : nb = mc : nd.$$

In a similar manner we may prove that

$$\frac{a}{m} : \frac{b}{n} = \frac{c}{m} : \frac{d}{n}.$$

Note. Either m or n may be unity.

307. *If three quantities are in continued proportion, the first is to the third as the square of the first is to the square of the second.*

Let $$a : b = b : c.$$

Then, $$\frac{a}{b} = \frac{b}{c}.$$

Therefore, $$\frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b}.$$

Or, $$\frac{a}{c} = \frac{a^2}{b^2}.$$

Whence, $$a : c = a^2 : b^2.$$

308. *If four quantities are in continued proportion, the first is to the fourth as the cube of the first is to the cube of the second.*

Let $$a : b = b : c = c : d.$$

Then, $$\frac{a}{b} = \frac{b}{c} = \frac{c}{d}.$$

Therefore, $$\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b}.$$

Or, $$\frac{a}{d} = \frac{a^3}{b^3}.$$

Whence, $$a : d = a^3 : b^3.$$

Note. The ratio $a^2 : b^2$ is called the *duplicate ratio*, and the ratio $a^3 : b^3$ the *triplicate ratio*, of $a : b$.

PROBLEMS.

309. **1.** Solve the equation,

$$x + 1 : x - 1 = a + b : a - b.$$

By Art. 301, $$2x : 2 = 2a : 2b.$$

Whence, by Art. 305, $x : 1 = a : b$.

Therefore, $$x = \frac{a}{b}, \text{ Ans.}$$

2. If $x : y = (x + z)^2 : (y + z)^2$, prove that z is a mean proportional between x and y.

From the given proportion, by Art. 293,

$$y(x + z)^2 = x(y + z)^2.$$

Or, $$x^2y + 2xyz + yz^2 = xy^2 + 2xyz + xz^2.$$

Or, $$x^2y - xy^2 = xz^2 - yz^2.$$

Dividing by $x - y$, $$xy = z^2.$$

Therefore z is a mean proportional between x and y.

3. Find the first term of the proportion whose last three terms are 18, 6, and 27.

4. Find the second term of the proportion whose first, third, and fourth terms are 4, 20, and 55.

5. Find a fourth proportional to $\frac{2}{3}$, $\frac{3}{5}$, and $\frac{2}{7}$.

6. Find a third proportional to $\frac{3}{4}$ and $\frac{5}{6}$.

7. Find a mean proportional between 8 and 18.

8. Find a mean proportional between 14 and 42.

9. Find a mean proportional between $2\frac{2}{9}$ and $\frac{5}{12}$.

Solve the following equations:

10. $2x-5:3x+2=x-1:7x+1.$

11. $x^2-4:x^2-9=x^2-5x+6:x^2+4x+3.$

12. $x+\sqrt{1-x^2}:x-\sqrt{1-x^2}=a+\sqrt{b^2-a^2}:a-\sqrt{b^2-a^2}.$

13. $\begin{cases} x:y=3:5. \\ x:4=15:y. \end{cases}$ 14. $\begin{cases} x+y:x-y=a+b:a-b. \\ x^2+y^2=a^2b^2(a^2+b^2). \end{cases}$

15. Find two numbers in the ratio of $2\frac{1}{2}$ to 2, such that when each is diminished by 5, they shall be in the ratio of $1\frac{1}{3}$ to 1.

16. Divide 50 into two parts such that the greater increased by 3 shall be to the less diminished by 3, as 3 to 2.

17. Divide 12 into two parts such that their product shall be to the sum of their squares as 3 to 10.

18. Find two numbers in the ratio of 4 to 9, such that 12 is a mean proportional between them.

19. The sum of two numbers is to their difference as 10 to 3, and their product is 364. What are the numbers?

20. If $a-b:b-c=b:c$, prove that b is a mean proportional between a and c.

21. If $5a+4b:9a+2b=5b+4c:9b+2c$, prove that b is a mean proportional between a and c.

22. If $(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d)$, prove that $a:b=c:d$.

23. If $ax-by:cx-dy=ay-bz:cy-dz$, prove that y is a mean proportional between x and z.

24. Find two numbers such that if 3 be added to each, they will be in the ratio of 4 to 3; and if 8 be subtracted from each, they will be in the ratio of 9 to 4.

25. There are two numbers whose product is 96, and the difference of their cubes is to the cube of their difference as 19 to 1. What are the numbers?

26. Divide \$564 between A, B, and C, so that A's share may be to B's in the ratio of 5 to 9, and B's share to C's in the ratio of 7 to 10.

27. A railway passenger observes that a train passes him, moving in the opposite direction, in 2 seconds; whereas, if it had been moving in the same direction with him, it would have passed him in 30 seconds. Compare the rates of the two trains.

28. Each of two vessels contains a mixture of wine and water. A mixture, consisting of equal measures from the two vessels, contains as much wine as water; and another mixture, consisting of four measures from the first vessel and one from the second, is composed of wine and water in the ratio of 2 to 3. Find the ratio of wine to water in each vessel.

29. Divide a into two parts such that the first increased by b shall be to the second diminished by b, as $a+3b$ is to $a-3b$.

XXVII. ARITHMETICAL PROGRESSION.

310. An **Arithmetical Progression** is a series of terms, each of which is derived from the preceding by adding a constant quantity called the *common difference.*

Thus 1, 3, 5, 7, 9, 11, ... is an increasing arithmetical progression, in which the common difference is 2.

12, 9, 6, 3, 0, -3, ... is a decreasing arithmetical progression, in which the common difference is -3.

311. *Given the first term, a, the common difference, d, and the number of terms, n, to find the last term, l.*

The progression is

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots$$

It will be observed that the coefficient of d in any term is one less than the number of the term. Hence, in the nth, or last term, the coefficient of d will be $n-1$. That is,

$$l=a+(n-1)d. \qquad \text{(I.)}$$

312. *Given the first term, a, the last term, l, and the number of terms, n, to find the sum of the series, S.*

$$S=a+(a+d)+(a+2d)+\cdots+(l-d)+l.$$

Writing the series in reverse order,

$$S=l+(l-d)+(l-2d)+\cdots+(a+d)+a.$$

Adding these equations, term by term,

$$2S=(a+l)+(a+l)+(a+l)+\cdots+(a+l)+(a+l)$$
$$=n(a+l).$$

Therefore, $$S=\frac{n}{2}(a+l). \qquad \text{(II.)}$$

313. Substituting in (II.) the value of l from (I.), we have

$$S=\frac{n}{2}[2a+(n-1)d].$$

EXAMPLES.

314. **1.** In the series $8, 5, 2, -1, -4, \dots$ to 27 terms, find the last term and the sum.

In this case, $a = 8$, $d = -3$, $n = 27$.

Substituting in (I.) and (II.),

$$l = 8 + (27 - 1)(-3) = 8 - 78 = -70.$$

$$S = \frac{27}{2}(8 - 70) = 27 \times (-31) = -837.$$

Note. The common difference may always be found by subtracting the first term from the second. Thus, in the series

$$\frac{5}{3}, -\frac{1}{6}, -2, \dots, \text{ we have } d = -\frac{1}{6} - \frac{5}{3} = -\frac{11}{6}.$$

In each of the following, find the last term and the sum of the series:

2. $1, 6, 11, \dots$ to 15 terms.

3. $7, 3, -1, \dots$ to 20 terms.

4. $-9, -6, -3, \dots$ to 23 terms.

5. $-5, -10, -15, \dots$ to 29 terms.

6. $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \dots$ to 35 terms.

7. $\frac{3}{5}, \frac{8}{15}, \dots$ to 19 terms.

8. $\frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \dots$ to 16 terms.

9. $\frac{1}{2}, \frac{5}{11}, \dots$ to 22 terms.

10. $-3, -\frac{5}{2}, \dots$ to 17 terms.

11. $-\frac{2}{5}, \frac{1}{3}, \dots$ to 14 terms.

315. If any three of the five elements of an arithmetical progression are given, the other two may be found by substituting the known values in the fundamental formulæ (I.) and (II.), and solving the resulting equations.

1. Given $a = -\frac{5}{3}$, $n = 20$, $S = -\frac{5}{3}$; find d and l.

Substituting the given values in (I.) and (II.), we have

$$l = -\frac{5}{3} + 19d. \qquad (1)$$

$$-\frac{5}{3} = 10\left(-\frac{5}{3} + l\right); \text{ or } -\frac{1}{6} = -\frac{5}{3} + l. \qquad (2)$$

From (2), $l = \frac{5}{3} - \frac{1}{6} = \frac{3}{2}$.

Substituting in (1),

$$\frac{3}{2} = -\frac{5}{3} + 19d; \text{ or } d = \frac{1}{6}.$$

Ans. $d = \frac{1}{6}$, $l = \frac{3}{2}$.

2. Given $d = -3$, $l = -39$, $S = -264$; find a and n.

Substituting in (I.) and (II.),

$$-39 = a + (n-1)(-3), \text{ or } a = 3n - 42. \qquad (1)$$

$$-264 = \frac{n}{2}(a - 39), \text{ or } an - 39n = -528. \qquad (2)$$

Substituting the value of a from (1) in (2),

$$3n^2 - 42n - 39n = -528,$$

or,

$$n^2 - 27n = -176.$$

Whence,

$$n = \frac{27 \pm \sqrt{729 - 704}}{2} = \frac{27 \pm 5}{2} = 16 \text{ or } 11.$$

Substituting in (1),

$$a = 48 - 42, \text{ or } 33 - 42 = 6 \text{ or } -9.$$

Ans. $a = 6$, $n = 16$; or, $a = -9$, $n = 11$.

Note. The interpretation of these answers is as follows:

If $a = 6$, and $n = 16$, the series is

$6, 3, 0, -3, -6, -9, -12, -15, -18, -21, -24, -27, -30, -33, -36, -39.$

If $a = -9$, and $n = 11$, the series is

$-9, -12, -15, -18, -21, -24, -27, -30, -33, -36, -39.$

In each of these results the last term is -39, and the sum -264.

3. Given $a = \frac{1}{3}$, $d = -\frac{1}{12}$, $S = -\frac{3}{2}$; find l and n.

Substituting in (I.) and (II.),

$$l = \frac{1}{3} + (n-1)\left(-\frac{1}{12}\right), \text{ or } l = \frac{5-n}{12}. \qquad (1)$$

$$-\frac{3}{2} = \frac{n}{2}\left(\frac{1}{3} + l\right), \text{ or } n + 3ln = -9. \qquad (2)$$

Substituting the value of l from (1) in (2),

$$n + \frac{5n - n^2}{4} = -9, \text{ or } n^2 - 9n = 36.$$

Solving this equation, $n = 12$ or -3.

The second value is inapplicable, as no significance can be attached to a negative number of terms.

Substituting the value of n in (1),

$$l = \frac{5-12}{12} = -\frac{7}{12}.$$

Ans. $l = -\frac{7}{12}$, $n = 12$.

Note. A negative or fractional value of n is always inapplicable, and should be rejected together with all other values dependent on it.

EXAMPLES.

4. Given $d = 4$, $l = 75$, $n = 19$; find a and S.

5. Given $d = -1$, $n = 15$, $S = -\frac{165}{2}$; find a and l.

6. Given $a=-\frac{2}{3}$, $n=18$, $l=5$; find d and S.

7. Given $a=-\frac{3}{4}$, $n=7$, $S=-7$; find d and l.

8. Given $a=\frac{3}{2}$, $l=-\frac{57}{2}$, $S=-\frac{351}{2}$; find d and n.

9. Given $l=-31$, $n=13$, $S=-169$; find a and d.

10. Given $d=-3$, $S=-328$, $a=2$; find l and n.

11. Given $a=3$, $l=42\frac{2}{3}$, $d=2\frac{1}{3}$; find n and S.

12. Given $d=-4$, $n=17$, $S=-493$; find a and l.

13. Given $l=\frac{7}{2}$, $d=\frac{1}{3}$, $S=20$; find a and n.

14. Given $l=\frac{79}{2}$, $n=21$, $S=\frac{819}{2}$; find a and d.

15. Given $a=-\frac{1}{3}$, $l=-\frac{4}{3}$, $S=-\frac{40}{3}$; find d and n.

16. Given $a=-\frac{3}{4}$, $n=15$, $S=120$; find d and l.

17. Given $l=-47$, $d=-1$, $S=-1118$; find a and n.

18. Given $a=6$, $d=-\frac{5}{3}$, $S=-\frac{203}{3}$; find n and l.

From (I.) and (II.) general formulæ for the solution of cases like the above may be readily derived.

19. Given a, d, and S; derive the formula for n.

Substituting the value of l from (I.) in (II.),

$$2S=n[2a+(n-1)d], \text{ or } dn^2+(2a-d)n=2S.$$

This is a quadratic in n, and may be solved by the method of Art. 265.

Multiplying by $4d$, and adding $(2a-d)^2$ to both members,

$$4d^2n^2+4d(2a-d)n+(2a-d)^2=8dS+(2a-d)^2.$$

Extracting the square root,

$$2dn+2a-d=\pm\sqrt{8dS+(2a-d)^2}.$$

Whence, $$n=\frac{d-2a\pm\sqrt{8dS+(2a-d)^2}}{2d}.$$

20. Given a, l, and n; derive the formula for d.

21. Given a, n, and S; derive the formulæ for d and l.

22. Given d, n, and S; derive the formulæ for a and l.

23. Given a, d, and l; derive the formulæ for n and S.

24. Given d, l, and n; derive the formulæ for a and S.

25. Given l, n, and S; derive the formulæ for a and d.

26. Given a, d, and S; derive the formula for l.

27. Given a, l, and S; derive the formulæ for d and n.

28. Given d, l, and S; derive the formulæ for a and n.

316. *To insert any number of arithmetical means between two given terms.*

1. Insert 5 arithmetical means between 3 and -5.

This signifies that we are to find 7 terms in arithmetical progression, such that the first term is 3, and the last term -5.

Substituting $a=3$, $l=-5$, and $n=7$ in (I.), we have

$$-5=3+6d, \text{ or } d=-\frac{4}{3}.$$

Hence, the required series is

$$3,\ \frac{5}{3},\ \frac{1}{3},\ -1,\ -\frac{7}{3},\ -\frac{11}{3},\ -5.$$

EXAMPLES.

2. Insert 5 arithmetical means between 2 and 4.

3. Insert 7 arithmetical means between 3 and -1.

4. Insert 4 arithmetical means between -1 and -7.

5. Insert 6 arithmetical means between -8 and -4.

6. Insert 8 arithmetical means between $\frac{1}{2}$ and $-\frac{13}{10}$.

Note. The arithmetical mean between two quantities, a and b, may be found as follows:

Let x denote the required mean; then, by the nature of the progression,
$$x - a = b - x, \text{ or } 2x = a + b.$$

Whence,
$$x = \frac{a+b}{2}.$$

That is, *the arithmetical mean between two quantities is equal to one-half their sum.*

7. Find the arithmetical mean between $\frac{7}{3}$ and $-\frac{9}{5}$.

8. Find the arithmetical mean between $(a+b)^2$ and $-(a-b)^2$.

9. Find the arithmetical mean between $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$.

PROBLEMS.

317. **1.** The sixth term of an arithmetical progression is $\frac{5}{6}$, and the fifteenth term is $\frac{16}{3}$. Find the first term.

By Art. 311, the sixth term is $a + 5d$, and the fifteenth term is $a + 14d$; hence,

$$\begin{cases} a + 5d = \frac{5}{6}. & (1) \\ a + 14d = \frac{16}{3}. & (2) \end{cases}$$

Subtracting (1) from (2), $9d = \frac{9}{2}$, or $d = \frac{1}{2}$.

Substituting in (2), $a + 7 = \frac{16}{3}$; whence, $a = -\frac{5}{3}$, *Ans.*

2. Find four quantities in arithmetical progression such that the product of the extremes shall be 45, and the product of the means 77.

Let the quantities be $x - 3y$, $x - y$, $x + y$, and $x + 3y$. Then, by the conditions,

$$\begin{cases} x^2 - 9y^2 = 45. \\ x^2 - y^2 = 77. \end{cases}$$

Solving these equations, $x = \pm 9$ and $y = \pm 2$.

Therefore the quantities are 3, 7, 11, and 15; or, -3, -7, -11, and -15.

Note. In problems like the above it is convenient to represent the unknown quantities by *symmetrical* expressions. Thus, if five quantities had been required, we should have represented them by $x - 2y$, $x - y$, x, $x + y$, and $x + 2y$.

3. Find the sum of the odd numbers from 1 to 100.

4. The seventh term of an arithmetical progression is 27, and the thirteenth term is -3. Find the twenty-first term.

5. Find four numbers in arithmetical progression such that the sum of the first and third shall be 22, and the sum of the second and fourth 36.

6. A person saves \$270 the first year, \$245 the second, and so on. In how many years will a person who saves every year \$145 have saved as much as he?

7. In the progression m, $2m - 3n$, $3m - 6n$, ... to 10 terms, find the last term and the sum of the series.

8. The seventh term of an arithmetical progression is $5a + 4b$, and the nineteenth term is $9a - 2b$. Find the fifteenth term.

9. Find the sum of the even numbers beginning with 2 and ending with 500.

10. The sum of the squares of the extremes of four numbers in arithmetical progression is 200, and the sum of the squares of the means is 136. What are the numbers?

11. The seventh term of an arithmetical progression is $-\frac{1}{2}$, the thirteenth term is $\frac{3}{2}$, and the last term is $\frac{9}{2}$. Find the number of terms.

12. Find five quantities in arithmetical progression such that the sum of the first, third, and fourth is 3, and the product of the second and fifth is -8.

13. Two persons start together. One travels 10 leagues a day; the other 8 leagues the first day, which he augments daily by half a league. After how many days, and at what distance from the point of departure, will they come together?

14. A body falls $16\frac{1}{12}$ feet the first second, and in each succeeding second $32\frac{1}{6}$ feet more than in the next preceding one. How far will it fall in 16 seconds?

15. Find three quantities in arithmetical progression such that the sum of the squares of the first and third exceeds the second by 123, and the second exceeds one-third the first by 6.

16. After A had travelled $2\frac{3}{4}$ hours at the rate of 4 miles an hour, B set out to overtake him, and went $4\frac{1}{2}$ miles the first hour, $4\frac{3}{4}$ the second, 5 the third, and so on, increasing his speed a quarter of a mile every hour. In how many hours would he overtake A?

17. If a person should save \$100 a year, and put this sum at simple interest at 5 per cent at the end of each year, to how much would his property amount at the end of 20 years?

18. The digits of a number of three figures are in arithmetical progression; the first digit exceeds the sum of the second and third by 1; and if 594 be subtracted from the number, the digits will be inverted. Find the number.

XXVIII. GEOMETRICAL PROGRESSION.

318. A **Geometrical Progression** is a series of terms, each of which is derived from the preceding by multiplying by a constant quantity called the *ratio*.

Thus, 2, 6, 18, 54, 162, ··· is an increasing geometrical progression in which the ratio is 3.

9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, ... is a decreasing geometrical progression in which the ratio is $\frac{1}{3}$.

Negative values of the ratio are also admissible; thus, $-3, 6, -12, 24, -48, \ldots$ is a progression in which the ratio is -2.

319. *Given the first term, a, the ratio, r, and the number of terms, n, to find the last term, l.*

The progression is $a, ar, ar^2, ar^3, \ldots$

It will be observed that the exponent of r in any term is one less than the number of the term. Hence, in the nth or last term, the exponent of r will be $n-1$. That is,

$$l = ar^{n-1}. \qquad \text{(I.)}$$

320. *Given the first term, a, the last term, l, and the ratio, r, to find the sum of the series, S.*

$$S = a + ar + ar^2 + \cdots + ar^{n-3} + ar^{n-2} + ar^{n-1}.$$

Multiplying each term by r,

$$rS = ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} + ar^n.$$

Subtracting the first equation from the second,

$$rS - S = ar^n - a; \text{ or, } S = \frac{ar^n - a}{r-1}.$$

But from (I.), Art. 319, $rl = ar^n$. Hence,

$$S = \frac{rl - a}{r - 1}. \qquad \text{(II.)}$$

EXAMPLES.

321. **1.** In the series $3, 1, \frac{1}{3}, \ldots$ to 7 terms, find the last term and the sum.

In this case, $a=3$, $r=\frac{1}{3}$, $n=7$. Substituting in (I.) and (II.),

$$l=3\left(\frac{1}{3}\right)^6=\frac{1}{3^5}=\frac{1}{243}.$$

$$S=\frac{\frac{1}{3}\times\frac{1}{243}-3}{\frac{1}{3}-1}=\frac{\frac{1}{729}-3}{-\frac{2}{3}}=\frac{-\frac{2186}{729}}{-\frac{2}{3}}=\frac{1093}{243}.$$

Note. The ratio may always be found by dividing the second term by the first.

2. In the series $-2, 6, -18, 54, \ldots$ to 8 terms, find the last term and the sum.

In this case, $a=-2$, $r=\frac{6}{-2}=-3$, $n=8$. Hence,

$$l=-2(-3)^7=-2\times(-2187)=4374.$$

$$S=\frac{-3\times 4374-(-2)}{-3-1}=\frac{-13122+2}{-4}=3280.$$

In each of the following find the last term and the sum of the series:

3. $1, 2, 4, \ldots$ to 9 terms.

4. $3, 2, \frac{4}{3}, \ldots$ to 7 terms.

5. $-2, 8, -32, \ldots$ to 6 terms.

6. $2, -1, \frac{1}{2}, \ldots$ to 10 terms.

7. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ to 11 terms.

8. $\frac{2}{3}, -1, \frac{3}{2}, \ldots$ to 8 terms.

9. 8, 4, 2, ... to 9 terms.

10. $\frac{3}{4}, -\frac{1}{4}, \frac{1}{12}, \ldots$ to 6 terms.

11. 3, -6, 12, ... to 7 terms.

12. $-\frac{2}{3}, -\frac{1}{3}, -\frac{1}{6}, \ldots$ to 10 terms.

322. If any three of the five elements of a geometrical progression are given, the other two may be found by substituting the known values in the fundamental formulæ (I.) and (II.), and solving the resulting equations.

But in certain cases the operation involves the solution of an equation of a degree higher than the second; and in others the unknown quantity appears as an exponent, the solution of which form of equation can usually only be effected by the use of logarithms (Art. 363). In all such examples in the present chapter, the equations may be solved by inspection.

1. Given $a = -2$, $n = 5$, $l = -32$; find r and S.

Substituting the given values in (I.), we have

$$-32 = -2r^4; \text{ whence, } r^4 = 16, \text{ or } r = \pm 2.$$

Substituting in (II.),

$$\text{If } r = 2,\ S = \frac{2(-32)-(-2)}{2-1} = -64 + 2 = -62.$$

$$\text{If } r = -2,\ S = \frac{(-2)(-32)-(-2)}{-2-1} = \frac{64+2}{-3} = -22.$$

Ans. $r = 2$, $S = -62$; or, $r = -2$, $S = -22$.

Note. The interpretation of these answers is as follows:

If $r = 2$, the series is $-2, -4, -8, -16, -32$, in which the sum is -62.

If $r = -2$, the series is $-2, 4, -8, 16, -32$, in which the sum is -22.

2. Given $a=3$, $r=-\frac{1}{3}$, $S=\frac{1640}{729}$; find n and l.

Substituting in (II.),

$$\frac{1640}{729}=\frac{-\frac{1}{3}l-3}{-\frac{1}{3}-1}=\frac{l+9}{4}.$$

Whence, $\quad l+9=\frac{6560}{729}$; or, $l=-\frac{1}{729}$.

Substituting in (I.),

$$-\frac{1}{729}=3\left(-\frac{1}{3}\right)^{n-1}; \text{ or, } \left(-\frac{1}{3}\right)^{n-1}=-\frac{1}{2187}.$$

Whence, by inspection,

$$n-1=7, \text{ or } n=8.$$

EXAMPLES.

3. Given $r=2$, $n=10$, $l=256$; find a and S.

4. Given $r=-2$, $n=6$, $S=\frac{63}{2}$; find a and l.

5. Given $a=2$, $n=7$, $l=1458$; find r and S.

6. Given $a=1$, $r=3$, $l=81$; find n and S.

7. Given $r=\frac{1}{3}$, $n=8$, $S=\frac{6560}{6561}$; find a and l.

8. Given $a=3$, $n=6$, $l=-\frac{3}{1024}$; find r and S.

9. Given $a=2$, $l=\frac{1}{32}$, $S=\frac{127}{32}$; find n and r.

10. Given $a=\frac{1}{2}$, $r=-3$, $S=-91$; find n and l.

11. Given $l=-128$, $r=2$, $S=-255$; find a and n.

From (I.) and (II.) general formulæ may be derived for the solution of cases like the above.

12. Given a, r, and S; derive the formula for l.

13. Given a, l, and S; derive the formula for r.

14. Given r, l, and S; derive the formula for a.

15. Given r, n, and l; derive the formulæ for a and S.

16. Given r, n, and S; derive the formulæ for a and l.

17. Given a, n, and l; derive the formulæ for r and S.

Note. If the given elements are n, l, and S, equations for a and r may be found, but there are no definite formulæ for their values. The same is the case when the given elements are a, n, and S.

The general formulæ for n involve logarithms; these cases are discussed in Art. 363.

323. The limit to which the sum of the terms of a *decreasing* geometrical progression approaches, as the number of terms increases indefinitely, is called the *sum of the series to infinity*.

The value of S in formula (II.) may be written

$$S=\frac{a-rl}{1-r}.$$

In a decreasing geometrical progression, the greater the number of terms taken, the smaller will be the value of the last term. Hence, as the number of terms increases indefinitely, the term rl decreases indefinitely and approaches the limit 0.

Therefore the fraction $\frac{a-rl}{1-r}$ approaches the limit $\frac{a}{1-r}$.

Hence, the sum of a decreasing geometrical progression to infinity is given by the formula

$$S=\frac{a}{1-r}. \qquad \text{(III.)}$$

EXAMPLES.

1. Find the sum of the series $4, -\frac{8}{3}, \frac{16}{9}, \dots$ to infinity.

In this case, $$a = 4,\ r = -\frac{2}{3}.$$

Substituting in (III.), $S = \dfrac{4}{1+\frac{2}{3}} = \dfrac{12}{5}$, *Ans.*

Find the sum of the following to infinity:

2. $2, 1, \frac{1}{2}, \dots$

3. $4, -2, 1, \dots$

4. $-1, \frac{1}{3}, -\frac{1}{9}, \dots$

5. $-3, -\frac{3}{5}, -\frac{3}{25}, \dots$

6. $\frac{3}{4}, \frac{1}{2}, \frac{1}{3}, \dots$

7. $3, -\frac{3}{10}, \frac{3}{100}, \dots$

8. $-8, -\frac{2}{5}, -\frac{1}{50}, \dots$

9. $1, -\frac{a^2}{x^2}, \frac{a^4}{x^4}, \dots$

324. *To find the value of a repeating decimal.*

This is a case of finding the sum of a geometrical progression to infinity, and may be solved by the formula of Art. 323.

1. Find the value of .85151 ⋯

$$.85151 \cdots = .8 + .051 + .00051 + \cdots$$

The terms after the first constitute a geometrical progression in which $a = .051$, and $r = .01$.

Substituting in (III.),

$$S = \frac{.051}{1 - .01} = \frac{.051}{.99} = \frac{51}{990} = \frac{17}{330}.$$

Hence the value of the given decimal is

$$\frac{8}{10} + \frac{17}{330} = \frac{281}{330},\ \textit{Ans.}$$

EXAMPLES.

Find the values of the following:

2. .7272…	**4.** .7333…	**6.** .110303…
3. .407407…	**5.** .52121…	**7.** .215454…

325. *To insert any number of geometrical means between two given terms.*

1. Insert 4 geometrical means between 2 and $\frac{64}{243}$.

This signifies that we are to find 6 terms in geometrical progression such that the first term is 2, and the last term $\frac{64}{243}$.

Substituting $a = 2$, $n = 6$, $l = \frac{64}{243}$ in (I.), we have

$$\frac{64}{243} = 2r^5; \text{ whence } r^5 = \frac{32}{243}, \text{ or } r = \frac{2}{3}.$$

Hence the required series is

$$2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \frac{32}{81}, \frac{64}{243}.$$

EXAMPLES.

2. Insert 6 geometrical means between 3 and $\frac{128}{729}$.

3. Insert 5 geometrical means between $\frac{1}{2}$ and $364\frac{1}{2}$.

4. Insert 6 geometrical means between -2 and -4374.

5. Insert 7 geometrical means between $\frac{3}{2}$ and $\frac{3}{512}$.

6. Insert 5 geometrical means between -2 and -128.

7. Insert 4 geometrical means between 3 and $-\frac{729}{1024}$.

Note. The geometrical mean between two quantities, a and b, may be found as follows:

Let x denote the required mean; then, by the nature of the progression,

$$\frac{x}{a}=\frac{b}{x}, \text{ or } x^2=ab.$$

Whence, $$x=\sqrt{ab}.$$

That is, *the geometrical mean between two quantities is equal to the square root of their product.*

8. Find the geometrical mean between $11\frac{2}{3}$ and $2\frac{1}{7}$.

9. Find the geometrical mean between $4x^2+12xy+9y^2$ and $4x^2-12xy+9y^2$.

10. Find the geometrical mean between $\dfrac{a^2-ab}{ab+b^2}$ and $\dfrac{a^2+ab}{ab-b^2}$.

PROBLEMS.

326. 1. Find three numbers in geometrical progression, such that their sum shall be 14 and the sum of their squares 84.

Let the quantities be a, ar, and ar^2; then, by the conditions,

$$\begin{cases} a+ar+ar^2 = 14. & (1) \\ a^2+a^2r^2+a^2r^4=84. & (2) \end{cases}$$

Dividing (2) by (1), $$a-ar+ar^2=6. \quad (3)$$

Subtracting (3) from (1), $$2ar=8, \text{ or } r=\frac{4}{a}. \quad (4)$$

Substituting in (1), $$a+4+\frac{16}{a}=14.$$

Or, $$a^2-10a=-16.$$

Solving this equation, $$a=8 \text{ or } 2.$$

Substituting in (4), $$r=\frac{4}{8} \text{ or } \frac{4}{2}=\frac{1}{2} \text{ or } 2.$$

Therefore, the numbers are 2, 4, and 8.

2. The fifth term of a geometrical progression is 48, and the eighth term is -384. Find the first term.

3. The sum of the first and second of four quantities in geometrical progression is 15, and the sum of the third and fourth is 60. What are the quantities?

4. Find three quantities in geometrical progression, such that the sum of the first and second is 20, and the third exceeds the second by 30.

5. The fourth term of a geometrical progression is -108, and the eighth term is -8748. Find the first term.

6. A person who saved every year half as much again as he saved the previous year, had in seven years saved \$2059. How much did he save the first year?

7. The elastic power of a ball, which falls from a height of a hundred feet, causes it to rise 0.9375 of the height from which it fell, and to continue in this way diminishing the height to which it will rise, in geometrical progression, until it comes to rest. How far will it have moved?

8. The sum of four quantities in geometrical progression is 30, and the quotient of the fourth quantity divided by the sum of the second and third is $\frac{4}{3}$. Find the quantities.

9. The third term of a geometrical progression is $\frac{1}{24}$, and the sixth term is $\frac{9}{512}$. Find the eighth term.

10. Divide the number 39 into three parts in geometrical progression, such that the third part shall exceed the first by 24.

11. The product of three numbers in geometrical progression is 64, and the sum of the squares of the first and third is 68. What are the numbers?

12. The product of three quantities in geometrical progression is 8, and the sum of their cubes is 73. What are the quantities?

XXIX. BINOMIAL THEOREM.

327. The **Binomial Theorem** is a formula by means of which a binomial may be raised to any required power without going through the process of actual multiplication.

Examples of its application have been given in Art. 196.

PROOF OF THE THEOREM FOR A POSITIVE INTEGRAL EXPONENT.

328. Assuming the laws of Art. 196 to hold for the expansion of $(a+x)^n$, where n is any positive integer:

The exponent of a in the first term is n, and decreases by 1 in each succeeding term.

The exponent of x in the second term is 1, and increases by 1 in each succeeding term.

The coefficient of the first term is 1; of the second term, n; multiplying the coefficient of the second term by $n-1$, the exponent of a in that term, and dividing the result by the exponent of x increased by 1, or 2, we have $\frac{n(n-1)}{2}$ as the coefficient of the third term; and so on.

Thus,
$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3 + \cdots$$

This result is called the *Binomial Theorem.*

329. To prove that it holds for any positive integral exponent, multiply both members by $a+x$. Then,

$$\begin{aligned}(a+x)^{n+1} = a^{n+1} &+ na^n x + \frac{n(n-1)}{1\cdot 2}a^{n-1}x^2 \\ &+ \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-2}x^3 + \cdots \\ &+ a^n x + na^{n-1}x^2 + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^3 + \cdots\end{aligned}$$

Collecting the terms which contain like powers of a and x, the result

$$= a^{n+1} + (n+1)a^n x + \left[\frac{n(n-1)}{1\cdot 2} + n\right] a^{n-1}x^2$$

$$+ \left[\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} + \frac{n(n-1)}{1\cdot 2}\right] a^{n-2}x^3 + \cdots$$

$$= a^{n+1} + (n+1)a^n x + n\left(\frac{n-1}{2} + 1\right) a^{n-1}x^2$$

$$+ \frac{n(n-1)}{1\cdot 2}\left(\frac{n-2}{3} + 1\right) a^{n-2}x^3 + \cdots$$

$$= a^{n+1} + (n+1)a^n x + n\left(\frac{n+1}{2}\right) a^{n-1}x^2$$

$$+ \frac{n(n-1)}{1\cdot 2}\left(\frac{n+1}{3}\right) a^{n-2}x^3 + \cdots$$

$$= a^{n+1} + (n+1)a^n x + \frac{(n+1)n}{1\cdot 2} a^{n-1}x^2$$

$$+ \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3} a^{n-2}x^3 + \cdots$$

It will be observed that the result is in the same form as the expansion of $(a+x)^n$, having $n+1$ in place of n.

Hence, if the theorem holds for any positive integral exponent, n, it also holds when that exponent is increased by 1.

But, in Art. 196, the theorem was shown to hold for $(a+x)^4$; hence it also holds for $(a+x)^5$; and since it holds for $(a+x)^5$, it also holds for $(a+x)^6$; and so on. Therefore the theorem holds when the exponent is any positive integer.

Note 1. The above method of proof is known as the *Method of Induction.*

Note 2. In place of the denominators $1\cdot 2$, $1\cdot 2\cdot 3$, etc., it is customary to write $\lfloor\underline{2}$, $\lfloor\underline{3}$, etc. The symbol $\lfloor\underline{n}$, read "factorial n," signifies the product of the natural numbers from 1 to n inclusive.

330. If x is negative, the terms in the expansion will be alternately positive and negative. Thus,

$$(a-x)^n = a^n - na^{n-1}x + \frac{n(n-1)}{\underline{|2}}a^{n-2}x^2 - \frac{n(n-1)(n-2)}{\underline{|3}}a^{n-3}x^3 + \cdots$$

331. If $a = 1$, we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^3 + \cdots$$

EXAMPLES.

332. 1. Expand $(m^{-\frac{1}{3}} - \sqrt{n})^5$.

Proceeding as in Art. 196, we have

$$\begin{aligned}(m^{-\frac{1}{3}} - \sqrt{n})^5 &= [(m^{-\frac{1}{3}}) - (n^{\frac{1}{2}})]^5 \\ &= (m^{-\frac{1}{3}})^5 - 5(m^{-\frac{1}{3}})^4(n^{\frac{1}{2}}) + 10(m^{-\frac{1}{3}})^3(n^{\frac{1}{2}})^2 \\ &\quad - 10(m^{-\frac{1}{3}})^2(n^{\frac{1}{2}})^3 + 5(m^{-\frac{1}{3}})(n^{\frac{1}{2}})^4 - (n^{\frac{1}{2}})^5 \\ &= m^{-\frac{5}{3}} - 5m^{-\frac{4}{3}}n^{\frac{1}{2}} + 10m^{-1}n - 10m^{-\frac{2}{3}}n^{\frac{3}{2}} + 5m^{-\frac{1}{3}}n^2 - n^{\frac{5}{2}},\end{aligned}$$

Ans.

Expand the following:

2. $(c^{\frac{2}{3}} + d^{-\frac{3}{4}})^4$.

3. $(m^{-\frac{1}{2}} - n^2)^5$.

4. $\left(\frac{x}{y} - \frac{3y}{x}\right)^3$.

5. $(x^m + 2y^n)^5$.

6. $(a^3 + 3\sqrt{x})^4$.

7. $\left(\frac{m}{n} - \sqrt{mn}\right)^5$.

8. $\left(\frac{\sqrt[3]{x}}{\sqrt[3]{y^2}} + \frac{\sqrt[3]{y}}{\sqrt[3]{x^2}}\right)^3$.

9. $\left(m^2 - \frac{n^3}{2}\right)^4$.

10. $(a^{\frac{1}{2}}b^{-\frac{1}{3}} - a^{-\frac{1}{2}}b^{\frac{1}{3}})^5$.

11. $(\sqrt{a^3} - 3\sqrt[3]{a})^4$.

12. $\left(\frac{2x}{\sqrt{y}} + \frac{y}{2\sqrt{x}}\right)^4$.

13. $\left(a^{-2} - \frac{1}{3}x^{\frac{1}{2}}\right)^6$.

14. $(x^{\frac{3}{5}}+3y^{-\frac{2}{5}})^5$.

15. $\left(\dfrac{a\sqrt{b}}{2x^{\frac{1}{2}}}-\dfrac{2\sqrt[3]{x}}{a^2b^{\frac{1}{2}}}\right)^3$.

16. $(3a^{-\frac{3}{4}}\sqrt{b}-b^{-\frac{1}{2}}\sqrt[4]{a})^4$.

17. $\left(\sqrt{\dfrac{a}{b}}+2\sqrt{\dfrac{b}{a}}\right)^6$.

A trinomial may be raised to any power by the Binomial Theorem if two of its terms be enclosed in a parenthesis and regarded as a single term. (Compare Art. 195.)

Expand the following:

18. $(1-x-x^2)^4$.

19. $(x^2+x-1)^4$.

20. $(1+2x-x^2)^4$.

21. $(1-x+x^2)^5$.

333. *To find the rth or general term in the expansion of* $(a+x)^n$.

The following laws will be observed to hold for any term in the expansion of $(a+x)^n$:

1. The exponent of x is less by 1 than the number of the term.

2. The exponent of a is n minus the exponent of x.

3. The last factor of the numerator is greater by 1 than the exponent of a.

4. The last factor of the denominator is the same as the exponent of x.

Therefore, in the rth term,

The exponent of x will be $r-1$.

The exponent of a will be $n-(r-1)$, or $n-r+1$.

The last factor of the numerator will be $n-r+2$.

The last factor of the denominator will be $r-1$.

Hence, the rth term

$$=\frac{n(n-1)(n-2)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}a^{n-r+1}x^{r-1}.$$

EXAMPLES.

1. Find the eighth term of $(3a^{\frac{1}{2}}-b^{-1})^{11}$.

In this case, $r=8$, $n=11$; hence, the eighth term

$$=\frac{11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}(3a^{\frac{1}{2}})^4(-b^{-1})^7$$

$$=330(81a^2)(-b^{-7})=-26730a^2b^{-7},\ \textit{Ans.}$$

Note. If the second term of the binomial is negative, it is convenient to enclose the term, sign and all, in a parenthesis, as shown in Ex. 1.

Or, the absolute value of the term may be found by the formula, and the sign determined in accordance with the principle that the odd terms of the expansion are positive, and the even terms negative.

Find the

2. Seventh term of $(a+x)^{11}$.

3. Sixth term of $(1+m)^{10}$.

4. Eighth term of $(c-d)^{12}$.

5. Fifth term of $(1-a^2)^{14}$.

6. Seventh term of $\left(\frac{a}{b}+\frac{b}{a}\right)^9$.

7. Fifth term of $(x-\sqrt{x})^{13}$.

8. Sixth term of $\left(a^{-3}-\frac{1}{2}ab\right)^9$.

9. Eighth term of $(x^{-1}+2y^{\frac{1}{2}})^{10}$.

10. Fourth term of $(a^{\frac{2}{3}}-3x^{-1})^{11}$.

11. Ninth term of $\left(\sqrt{m}+\frac{2}{\sqrt[4]{m}}\right)^{12}$.

XXX. LOGARITHMS.

334. Every positive number may be expressed, exactly or approximately, as a power of 10; thus,

$$100 = 10^2;\ 13 = 10^{1.1139\ldots};\ \text{etc.}$$

When thus expressed, the corresponding exponent is called its **Logarithm to the base 10**; thus, 2 is the logarithm of 100 to the base 10, a relation which is written

$$\log_{10} 100 = 2, \text{ or simply } \log 100 = 2.$$

And, in general, if $10^x = m$, then $x = \log m$.

335. Any positive number except unity may be taken as the base of a system of logarithms; thus, if $a^x = m$, x is the logarithm of m to the base a.

Logarithms to the base 10 are called *Common Logarithms*, and are the only ones used for numerical computations. If no base is expressed, the base 10 is understood.

336. By Arts. 220 and 221, we have

$$10^0 = 1, \qquad 10^{-1} = \frac{1}{10} = .1,$$

$$10^1 = 10, \qquad 10^{-2} = \frac{1}{100} = .01,$$

$$10^2 = 100, \qquad 10^{-3} = \frac{1}{1000} = .001, \text{ etc.}$$

Whence, by Art. 334,

$$\log 1 = 0, \qquad \log .1 = -1 = 9 - 10,$$

$$\log 10 = 1, \qquad \log .01 = -2 = 8 - 10,$$

$$\log 100 = 2, \qquad \log .001 = -3 = 7 - 10, \text{ etc.}$$

Note. The second form of the results for log .1, log .01, etc., is preferable in practice.

337. It is evident from the preceding article that the logarithm of a number greater than 1 is positive, and the logarithm of a number less than 1, and greater than 0, is negative.

338. If a number is not an exact power of 10, its common logarithm can only be expressed approximately; the integral part of the logarithm is called the *characteristic*, and the decimal part the *mantissa*.

For example, $\log 13 = 1.1139.$

In this case the characteristic is 1, and the mantissa is .1139.

339. It is evident from the first column of Art. 336 that the logarithm of any number between

1 and 10 is equal to 0 plus a decimal;
10 and 100 is equal to 1 plus a decimal;
100 and 1000 is equal to 2 plus a decimal; etc.

Hence, the characteristic of the logarithm of a number, with *one* figure to the left of its decimal point, is 0; with *two* figures to the left of the decimal point, is 1; with *three* figures to the left of the decimal point, is 2; etc.

340. Similarly, from the second column of Art. 336, the logarithm of a decimal between

1 and .1 is equal to 9 plus a decimal -10;
.1 and .01 is equal to 8 plus a decimal -10;
.01 and .001 is equal to 7 plus a decimal -10; etc.

Hence, the characteristic of the logarithm of a decimal, with *no* ciphers between its decimal point and first significant figure, is 9, with -10 after the mantissa; of a decimal with *one* cipher between its point and first figure, is 8, with -10 after the mantissa; of a decimal with *two* ciphers between its point and first figure, is 7, with -10 after the mantissa; etc.

341. For reasons which will be given hereafter, only the mantissa of the logarithm is given in the table; the characteristic must be supplied by the reader. The rules for characteristic are based on Arts. 339 and 340.

I. *If the number is greater than* 1, *the characteristic is one less than the number of places to the left of the decimal point.*

II. *If the number is less than* 1, *subtract the number of ciphers between the decimal point and first significant figure from* 9, *writing* -10 *after the mantissa.*

Thus, characteristic of $\log 906328.5 = 5$;

characteristic of $\log .00702 \quad = 7$, with -10 after the mantissa.

Note. Some writers, in dealing with the characteristics of the logarithms of numbers less than 1, combine the two portions of the characteristic, writing the result as a *negative characteristic* before the mantissa. Thus, instead of $7.6036-10$, the student will frequently find $\bar{3}.6036$; a minus sign being written over the characteristic to denote that it alone is negative, the mantissa being always positive.

PROPERTIES OF LOGARITHMS.

342. *The logarithm of a product is equal to the sum of the logarithms of its factors.*

Assume the equations

$$\left.\begin{array}{l}10^x = m\\10^y = n\end{array}\right\}; \text{ whence, by Art. 334, } \left\{\begin{array}{l}x = \log m.\\y = \log n.\end{array}\right.$$

Multiplying, $\quad 10^x \times 10^y = mn$, or $10^{x+y} = mn$.

Whence, $\quad \log mn = x + y$.

Substituting the values of x and y,

$$\log mn = \log m + \log n.$$

In a similar manner the theorem may be proved for the product of three or more factors.

343. By aid of the theorem of Art. 342, the logarithm of any composite number may be found when the logarithms of its factors are known.

1. Given $\log 2 = .3010$, $\log 3 = .4771$; find $\log 72$.

$$\begin{aligned}\log 72 &= \log(2 \times 2 \times 2 \times 3 \times 3)\\ &= \log 2 + \log 2 + \log 2 + \log 3 + \log 3\\ &= 3 \times \log 2 + 2 \times \log 3\\ &= .9030 + .9542\\ &= 1.8572, \textit{ Ans.}\end{aligned}$$

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 5 = .6990$, $\log 7 = .8451$; find the values of the following:

2. $\log 6$.
3. $\log 14$.
4. $\log 8$.
5. $\log 12$.
6. $\log 15$.
7. $\log 21$.
8. $\log 63$.
9. $\log 56$.
10. $\log 84$.
11. $\log 45$.
12. $\log 98$.
13. $\log 105$.
14. $\log 112$.
15. $\log 144$.
16. $\log 216$.
17. $\log 135$.
18. $\log 168$.
19. $\log 147$.
20. $\log 375$.
21. $\log 343$.

344. *The logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.*

Assume the equations

$$\left.\begin{array}{l}10^x = m\\ 10^y = n\end{array}\right\}; \text{ whence, } \left\{\begin{array}{l}x = \log m.\\ y = \log n.\end{array}\right.$$

Dividing, $\dfrac{10^x}{10^y} = \dfrac{m}{n}$, or $10^{x-y} = \dfrac{m}{n}$.

Whence, $$\log \frac{m}{n} = x - y.$$

Substituting the values of x and y,

$$\log \frac{m}{n} = \log m - \log n.$$

345. 1. Given $\log 2 = .3010$; find $\log 5$.

$$\log 5 = \log \frac{10}{2} = \log 10 - \log 2$$
$$= 1 - .3010 = .6990, \quad \textit{Ans.}$$

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 7 = .8451$; find the values of the following:

2. $\log \frac{7}{3}$.
3. $\log \frac{10}{7}$.
4. $\log 3\frac{1}{3}$.
5. $\log 35$.
6. $\log \frac{21}{16}$.
7. $\log 125$.
8. $\log \frac{42}{25}$.
9. $\log 175$.
10. $\log 11\frac{1}{9}$.
11. $\log 7\frac{1}{7}$.
12. $\log \frac{35}{6}$.
13. $\log 5\frac{4}{9}$.

346. *The logarithm of any power of a quantity is equal to the logarithm of the quantity multiplied by the exponent of the power.*

Assume the equation

$$10^x = m; \quad \text{whence, } x = \log m.$$

Raising both members to the pth power,

$$10^{px} = m^p; \quad \text{whence, } \log m^p = px = p \log m.$$

347. *The logarithm of any root of a quantity is equal to the logarithm of the quantity divided by the index of the root.*

For, $\quad \log \sqrt[r]{m} = \log (m^{\frac{1}{r}}) = \frac{1}{r} \log m. \quad$ (Art. 346.)

348. 1. Given $\log 2 = .3010$; find the logarithm of $2^{\frac{5}{3}}$.

$$\log 2^{\frac{5}{3}} = \frac{5}{3} \times \log 2 = \frac{5}{3} \times .3010 = .5017, \quad \textit{Ans.}$$

Note. To multiply a logarithm by a fraction, multiply first by the numerator, and divide the result by the denominator.

2. Given $\log 3 = .4771$; find the logarithm of $\sqrt[8]{3}$.

$$\log \sqrt[8]{3} = \frac{\log 3}{8} = \frac{.4771}{8} = .0596, \textit{ Ans.}$$

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 7 = .8451$; find the values of the following:

3. $\log 3^{\frac{3}{5}}$.	**7.** $\log 12^{\frac{2}{3}}$.	**11.** $\log 15^{\frac{5}{6}}$.	**15.** $\log \sqrt[6]{5}$.
4. $\log 2^9$.	**8.** $\log 21^{\frac{1}{2}}$.	**12.** $\log \sqrt{7}$.	**16.** $\log \sqrt[4]{35}$.
5. $\log 7^5$.	**9.** $\log 14^4$.	**13.** $\log \sqrt[3]{3}$.	**17.** $\log \sqrt[9]{98}$.
6. $\log 5^{\frac{1}{5}}$.	**10.** $\log 25^{\frac{7}{3}}$.	**14.** $\log \sqrt[7]{2}$.	**18.** $\log \sqrt[12]{126}$.

19. Find the logarithm of $\left(2^{\frac{1}{3}} \times 3^{\frac{5}{4}}\right)$.

By Art. 342, $\log\left(2^{\frac{1}{3}} \times 3^{\frac{5}{4}}\right) = \log 2^{\frac{1}{3}} + \log 3^{\frac{5}{4}}$

$$= \frac{1}{3}\log 2 + \frac{5}{4}\log 3 = .6967, \textit{ Ans.}$$

Find the values of the following:

20. $\log\left(\frac{10}{3}\right)^5$.	**22.** $\log\left(3^{\frac{1}{6}} \times 2^{\frac{3}{5}}\right)$.	**24.** $\log\sqrt{\frac{7}{3}}$.	**26.** $\log\sqrt[3]{\frac{28}{5}}$.
21. $\log\frac{7^{\frac{3}{4}}}{5^{\frac{2}{3}}}$.	**23.** $\log 3\sqrt[4]{7}$.	**25.** $\log\frac{\sqrt[3]{7}}{\sqrt[5]{2}}$.	**27.** $\log\frac{\sqrt{42}}{10^{\frac{2}{3}}}$.

349. *The mantissæ of the logarithms of numbers having the same sequence of figures are the same.*

To illustrate, suppose that $\log 3.053 = .4847$.

Then,
$$\begin{aligned}
\log 30.53 &= \log(10 \times 3.053) &&= \log 10 + \log 3.053\\
&= 1 + .4847 &&= 1.4847.\\
\log 305.3 &= \log(100 \times 3.053) &&= \log 100 + \log 3.053\\
&= 2 + .4847 &&= 2.4847.\\
\log .03053 &= \log(.01 \times 3.053) &&= \log .01 + \log 3.053\\
&= 8 - 10 + .4847 &&= 8.4847 - 10;\ \text{etc.}
\end{aligned}$$

It is evident from the above that, if a number be multiplied or divided by any integral power of 10, thus producing another number with the same sequence of figures, the mantissæ of their logarithms will be the same.

Thus, if $\log 3.053 = .4847$,

then, $\log 30.53 = 1.4847$, $\log .3053 = 9.4847-10$,

$\log 305.3 = 2.4847$, $\log .03053 = 8.4847-10$,

$\log 3053. = 3.4847$, $\log .003053 = 7.4847-10$, etc.

Note. The reason will now be seen for the statement made in Art. 341, that only the mantissæ are given in the table. For, to find the logarithm of any number, we have only to take from the table the mantissa corresponding to its sequence of figures, and the characteristic may then be prefixed in accordance with the rules of Art. 341.

This property of logarithms is only enjoyed by the common system, and constitutes its superiority over all others for numerical computations.

350. 1. Given $\log 2 = .3010$, $\log 3 = .4771$; find $\log .00432$.

$$\begin{aligned}\log 432 &= \log(2^4 \times 3^3)\\ &= 4\log 2 + 3\log 3\\ &= 1.2040 + 1.4313 = 2.6353.\end{aligned}$$

Whence, by Art. 341, $\log .00432 = 7.6353 - 10$, *Ans.*

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 7 = .8451$; find the values of the following:

2. $\log 1.8$.

3. $\log 2.25$.

4. $\log .196$.

5. $\log .048$.

6. $\log 38.4$.

7. $\log .0054$.

8. $\log .000315$.

9. $\log 7350$.

10. $\log 4.05$.

11. $\log .448$.

12. $\log 302.4$.

13. $\log .06174$.

14. $\log (8.1)^7$.

15. $\log \sqrt[5]{9.6}$.

16. $\log \sqrt[3]{1.62}$.

17. $\log (22.4)^{\frac{1}{8}}$.

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	414	453	492	531	569	607	645	682	719	755
12	792	828	864	899	934	969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	335	367	399	430
14	461	492	523	553	584	614	644	673	703	732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	279
17	304	330	355	380	405	430	455	480	504	529
18	553	577	601	625	648	672	695	718	742	765
19	788	810	833	856	878	900	923	945	967	989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	222	243	263	284	304	324	345	365	385	404
22	424	444	464	483	502	522	541	560	579	598
23	617	636	655	674	692	711	729	747	766	784
24	802	820	838	856	874	892	909	927	945	962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	183	200	216	232	249	265	281	298
27	314	330	346	362	378	393	409	425	440	456
28	472	487	502	518	533	548	564	579	594	609
29	624	639	654	669	683	698	713	728	742	757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	914	928	942	955	969	983	997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	145	159	172
33	185	198	211	224	237	250	263	276	289	302
34	315	328	340	353	366	378	391	403	416	428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	563	575	587	599	611	623	635	647	658	670
37	682	694	705	717	729	740	752	763	775	786
38	798	809	821	832	843	855	866	877	888	899
39	911	922	933	944	955	966	977	988	999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	128	138	149	160	170	180	191	201	212	222
42	232	243	253	263	274	284	294	304	314	325
43	335	345	355	365	375	385	395	405	415	425
44	435	444	454	464	474	484	493	503	513	522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	628	637	646	656	665	675	684	693	702	712
47	721	730	739	749	758	767	776	785	794	803
48	812	821	830	839	848	857	866	875	884	893
49	902	911	920	928	937	946	955	964	972	981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	093	101	110	118	126	135	143	152
52	160	168	177	185	193	202	210	218	226	235
53	243	251	259	267	275	284	292	300	308	316
54	324	332	340	348	356	364	372	380	388	396
No.	0	1	2	3	4	5	6	7	8	9

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	482	490	497	505	513	520	528	536	543	551
57	559	566	574	582	589	597	604	612	619	627
58	634	642	649	657	664	672	679	686	694	701
59	709	716	723	731	738	745	752	760	767	774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	853	860	868	875	882	889	896	903	910	917
62	924	931	938	945	952	959	966	973	980	987
63	993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	069	075	082	089	096	102	109	116	122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	195	202	209	215	222	228	235	241	248	254
67	261	267	274	280	287	293	299	306	312	319
68	325	331	338	344	351	357	363	370	376	382
69	388	395	401	407	414	420	426	432	439	445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	513	519	525	531	537	543	549	555	561	567
72	573	579	585	591	597	603	609	615	621	627
73	633	639	645	651	657	663	669	675	681	686
74	692	698	704	710	716	722	727	733	739	745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	808	814	820	825	831	837	842	848	854	859
77	865	871	876	882	887	893	899	904	910	915
78	921	927	932	938	943	949	954	960	965	971
79	976	982	987	993	998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	085	090	096	101	106	112	117	122	128	133
82	138	143	149	154	159	165	170	175	180	186
83	191	196	201	206	212	217	222	227	232	238
84	243	248	253	258	263	269	274	279	284	289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	345	350	355	360	365	370	375	380	385	390
87	395	400	405	410	415	420	425	430	435	440
88	445	450	455	460	465	469	474	479	484	489
89	494	499	504	509	513	518	523	528	533	538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	590	595	600	605	609	614	619	624	628	633
92	638	643	647	652	657	661	666	671	675	680
93	685	689	694	699	703	708	713	717	722	727
94	731	736	741	745	750	754	759	763	768	773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	823	827	832	836	841	845	850	854	859	863
97	868	872	877	881	886	890	894	899	903	908
98	912	917	921	926	930	934	939	943	948	952
99	956	961	965	969	974	978	983	987	991	996
No.	0	1	2	3	4	5	6	7	8	9

USE OF THE TABLE.

351. The table (pages 294 and 295) gives the mantissæ of the logarithms of all numbers from 100 to 1000, calculated to four decimal places.

352. *To find the logarithm of any number of three figures.*

Find in the column headed "No." the first two figures of the given number. Then the mantissa required will be found in the corresponding horizontal line, in the vertical column headed by the third figure of the number.

Finally, prefix the characteristic by the rules of Art. 339.

For example, $\log 168 = 2.2253.$

If only the last three figures of the mantissa are found, the first figure may be obtained from the nearest mantissa above, in the same column, consisting of four figures.

Thus, $\log .344 = 9.5366 - 10.$

353. For numbers of one or two figures, the column headed 0 may be used; for, by Art. 349, $\log 83$ has the same mantissa as $\log 830$, and $\log 9$ the same mantissa as $\log 900$.

Thus, $\log 83 = 1.9191$, and $\log 9 = 0.9542.$

354. *To find the logarithm of a number of more than three figures.*

For example, required the logarithm of 327.6.

We find from the table, $\log 327 = 2.5145.$

$$\log 328 = 2.5159.$$

That is, an increase of one unit in the number produces an increase of .0014 in the logarithm. Therefore an increase of .6 of a unit in the number will produce an increase of $.6 \times .0014$ in the logarithm, or .0008 to the nearest fourth decimal place. Hence,

$$\log 327.6 = 2.5145 + .0008 = 2.5153.$$

Note. The difference in the table between any mantissa and the mantissa of the next higher number of three figures, is called the *tabular difference.* It may always be obtained mentally.

The following rule is derived from the above:

Find from the table the mantissa of the first three significant figures, and the tabular difference.

Multiply the latter by the remaining figures of the number, with a decimal point before them.

Add the result to the mantissa of the first three figures, and prefix the proper characteristic.

355. **1.** Find the logarithm of .021508.

$$\text{Mantissa of } 215 = 3324$$

$$\begin{array}{lr} \text{Tabular difference} = & 21 \\ & \underline{.08} \\ \text{Correction} = & 1.68 = 2 \text{ nearly.} \end{array} \qquad \begin{array}{r} 3324 \\ \underline{2} \\ 3326 \end{array}$$

Ans. 8.3326 − 10.

EXAMPLES.

Find the logarithms of the following:

2. 80.	**6.** .7723.	**10.** 20.08.	**14.** 5.1809.
3. 6.3.	**7.** 1056.	**11.** 92461.	**15.** 1036.5.
4. 298.	**8.** 3.294.	**12.** .40322.	**16.** .086676.
5. .902.	**9.** .05205.	**13.** .007178.	**17.** .11507.

356. *To find the number corresponding to a logarithm.*

1. Required the number whose logarithm is 1.6571.

Find in the table the mantissa 6571. In the corresponding line, in the column headed "No.," we find 45, the first two figures of the required number, and at the head of the column we find 4, the third figure.

Since the characteristic is 1, there must be two figures to the left of the decimal point (Art. 339). Hence,

Number corresponding to 1.6571 = 45.4.

2. Required the number whose logarithm is 2.3934.

We find in the table the mantissa 3927, of which the corresponding number is 247, and the mantissa 3945, of which the corresponding number is 248.

That is, an increase of 18 in the mantissa produces an increase of one unit in the number corresponding. Therefore, an increase of 7 in the mantissa will produce an increase of $\frac{7}{18}$ of a unit in the number, or .39 nearly. Hence,

$$\text{Number corresponding} = 247 + .39 = 247.39.$$

We derive the following rule from the above operation:

Find from the table the next less mantissa, the three figures corresponding, and the tabular difference.

Subtract the next less from the given mantissa, and divide the remainder by the tabular difference.

Annex the quotient to the first three figures of the number, and point off the result.

357. 1. Find the number whose logarithm is $7.5264 - 10$.

5264
Next less mantissa = 5263 ; three figures corresponding = 336.
Tabular difference = 13)1.000(.077 = .08 nearly.
91
90

Since the characteristic is $7 - 10$, there must be two ciphers between the decimal point and first significant figure (Art. 339). Hence,

$$\text{Number corresponding} = .0033608, \ \textit{Ans.}$$

Note. In computations with four-place logarithms, the results cannot usually be depended upon to more than *four* significant figures.

EXAMPLES.

Find the numbers corresponding to the following logarithms:

2. 1.8055.

3. 9.4487 — 10.

4. 0.2165.

5. 3.9487.

6. 2.7364.

7. 8.1648 — 10.

8. 7.5209 — 10.

9. 4.0095.

10. 0.9774.

11. 9.3178 — 10.

12. 1.6482.

13. 7.0450 — 10.

14. 4.8016.

15. 8.1144 — 10.

16. 2.7015.

APPLICATIONS.

358. The value of an arithmetical quantity, in which the operations indicated involve only multiplication, division, involution, or evolution, may be most conveniently found by logarithms.

The utility of the process consists in the fact that addition takes the place of multiplication, subtraction of division, multiplication of involution, and division of evolution.

In operations with negative characteristics the rules of Algebra must be followed.

1. Find the value of $.0631 \times 7.208 \times .51272$.

By Art. 342, $\log(.0631 \times 7.208 \times .51272)$

$$= \log .0631 + \log 7.208 + \log .51272.$$

$$\begin{aligned} \log\ .0631 &= 8.8000 - 10 \\ \log\ 7.208 &= 0.8578 \\ \log .51272 &= \underline{9.7099 - 10} \end{aligned}$$

Adding, $\therefore \log \text{ of result} = 19.3677 - 20$

$$= 9.3677 - 10 \text{ (see Note 1).}$$

Number corresponding to $9.3677 - 10 = .2332$, *Ans.*

Note 1. If the sum is a negative logarithm, it should be reduced so that the negative part of the characteristic may be -10.

Thus, $19.3677 - 20$ is reduced to $9.3677 - 10$.

2. Find the value of $\dfrac{336.8}{7984}$.

By Art. 344, $\log\dfrac{336.8}{7984} = \log 336.8 - \log 7984.$

$$\log 336.8 = 12.5273 - 10 \quad \text{(see Note 2).}$$
$$\log\ 7984 = 3.9022$$

Subtracting, $\therefore \log$ of result $= 8.6251 - 10$

Number corresponding $= .04218$, *Ans.*

Note 2. To subtract a greater logarithm from a less, or to subtract a negative logarithm from a positive, increase the characteristic of the minuend by 10, writing -10 after the mantissa to compensate.

Thus, to subtract 3.9022 from 2.5273, write the minuend in the form $12.5273 - 10$; subtracting 3.9022 from this, the result is $8.6251 - 10$.

3. Find the value of $(.07396)^5$.

By Art. 346, $\log(.07396)^5 = 5 \times \log .07396.$

$$\log .07396 = 8.8690 - 10$$
$$5$$
$$44.3450 - 50$$
$$= 4.3450 - 10$$
$$= \log .000002213, \textit{ Ans.}$$

4. Find the value of $\sqrt[3]{.035063}$.

By Art. 347, $\log\sqrt[3]{.035063} = \frac{1}{3}\log .035063.$

$$\log .035063 = 8.5449 - 10$$
$$20. \qquad - 20 \quad \text{(see Note 3).}$$
$$3\overline{)28.5449 - 30}$$
$$9.5150 - 10$$
$$= \log .3274, \textit{ Ans.}$$

Note 3. To divide a negative logarithm, add to both parts such a multiple of 10 as will make the quantity standing after the mantissa exactly divisible by the divisor, with -10 as the quotient.

Thus, to divide $8.5449 - 10$ by 3, add 20 to both parts of the logarithm, giving the result $28.5449 - 30$. Dividing this by 3, the quotient is $9.5150 - 10$.

ARITHMETICAL COMPLEMENT.

359. The *Arithmetical Complement* of the logarithm of a number, or, briefly, the *cologarithm* of the number, is the logarithm of the reciprocal of that number.

Thus, $\text{colog } 409 = \log\frac{1}{409} = \log 1 - \log 409.$

$$\begin{aligned} \log 1 &= 10. \qquad -10 \quad \text{(Note 2, Art. 358.)} \\ \log 409 &= \underline{\;2.6117\qquad\;} \\ \therefore \text{colog } 409 &= 7.3883 - 10. \end{aligned}$$

$$\text{colog } .067 = \log\frac{1}{.067} = \log 1 - \log .067.$$

$$\begin{aligned} \log 1 &= 10. \qquad -10 \\ \log .067 &= \underline{8.8261 - 10} \\ \therefore \text{colog } .067 &= 1.1739. \end{aligned}$$

Note. The cologarithm may be calculated mentally from the logarithm by subtracting the last *significant* figure from 10, and each of the others from 9.

360. *Example.* Find the value of $\dfrac{.51384}{8.709 \times .0946}$.

$$\begin{aligned} \log\frac{.51384}{8.709 \times .0946} &= \log\left(.51384 \times \frac{1}{8.709} \times \frac{1}{.0946}\right) \\ &= \log .51384 + \log\frac{1}{8.709} + \log\frac{1}{.0946} \\ &= \log .51384 + \text{colog } 8.709 + \text{colog } .0946. \end{aligned}$$

$$\begin{aligned} \log .51384 &= 9.7109 - 10 \\ \text{colog } 8.709 &= 9.0601 - 10 \\ \text{colog } .0946 &= \underline{1.0241\qquad\quad} \\ & \;\; 9.7951 - 10 = \log .6239, \;\textit{Ans.} \end{aligned}$$

It is evident from the above that the logarithm of a fraction is equal to the logarithm of the numerator *plus* the cologarithm of the denominator.

Or, in general, to find the logarithm of a fraction whose terms are composed of factors,

Add together the logarithms of the factors of the numerator, and the cologarithms of the factors of the denominator.

Note. The value of the above fraction may be found without using cologarithms, as follows:

$$\log \frac{.51384}{8.709 \times .0946} = \log .51384 - \log (8.709 \times .0946)$$
$$= \log .51384 - (\log 8.709 + \log .0946).$$

The advantage of the use of cologarithms is that it exhibits the written work of computation in a more compact form.

EXAMPLES.

361. Note. A negative quantity can have no common logarithm, as is evident from the definition of Art. 334. If negative quantities occur in computation, they may be treated as if they were positive, and the *sign* of the result determined irrespective of the logarithmic work. See Ex. 34.

Find by logarithms the values of the following:

1. $9.238 \times .9152$.
2. $130.36 \times .08237$.
3. $721.3 \times (-3.0528)$.
4. $(-4.3264) \times (-.050377)$.
5. $.27031 \times .042809$.
6. $(-.063165) \times 11.134$.
7. $\dfrac{401.8}{52.37}$.
8. $\dfrac{7.2321}{10.813}$.
9. $\dfrac{-.3384}{.08659}$.
10. $\dfrac{9.163}{.0051422}$.
11. $\dfrac{22518}{64327}$.
12. $\dfrac{.007514}{-.015822}$.
13. $\dfrac{3.3681}{12.853 \times .6349}$.
14. $\dfrac{15.008 \times (-.0843)}{.06376 \times 4.248}$.
15. $\dfrac{(-2563) \times .03442}{714.8 \times (-.511)}$.
16. $\dfrac{121.6 \times (-9.025)}{(-48.3) \times 3662 \times (-.0856)}$.

17. $(23.86)^3$.	**22.** $(.8)^{\frac{2}{5}}$.	**27.** $\sqrt[5]{-3}$.
18. $(.532)^8$.	**23.** $(-3.16)^{\frac{4}{3}}$.	**28.** $\sqrt{.4294}$.
19. $(-1.0246)^7$.	**24.** $(.021)^{\frac{5}{2}}$.	**29.** $\sqrt[3]{.02305}$.
20. $(.09323)^5$.	**25.** $\sqrt{2}$.	**30.** $\sqrt[8]{1000}$.
21. $5^{\frac{2}{3}}$.	**26.** $\sqrt[4]{5}$.	**31.** $\sqrt[7]{-.00951}$.

32. $\sqrt[5]{.0001011}$.

33. Find the value of $\dfrac{2\sqrt[3]{5}}{3^{\frac{5}{6}}}$.

$$\log\frac{2\sqrt[3]{5}}{3^{\frac{5}{6}}} = \log 2 + \log\sqrt[3]{5} + \text{colog}\, 3^{\frac{5}{6}} \quad \text{(Art. 360.)}$$

$$= \log 2 + \frac{1}{3}\log 5 + \frac{5}{6}\,\text{colog}\, 3.$$

$$\log 2 = .3010$$

$$\log 5 = .6990; \qquad \text{divide by } 3 = .2330$$

$$\text{colog}\, 3 = 9.5229 - 10; \text{ multiply by } \frac{5}{6} = \underline{9.6024 - 10}$$

$$.1364$$

$$= \log 1.369, \; \textit{Ans.}$$

34. Find the value of $\sqrt[3]{\dfrac{-.03296}{7.962}}$.

$$\log\sqrt[3]{\frac{.03296}{7.962}} = \frac{1}{3}\log\frac{.03296}{7.962} = \frac{1}{3}(\log .03296 - \log 7.962)$$

$$\log .03296 = 8.5180 - 10$$

$$\log 7.962 = 0.9010$$

$$3\,)\overline{27.6170 - 30}$$

$$9.2057 - 10 = \log .1606.$$

Ans. $-.1606$.

Find the values of the following:

35. $2^{\frac{3}{2}} \times 3^{\frac{2}{3}}$.

36. $\dfrac{3^{\frac{5}{8}}}{4^{\frac{2}{3}}}$.

37. $\dfrac{5^{\frac{3}{7}}}{(-10)^{\frac{2}{9}}}$.

38. $\left(\dfrac{6}{7}\right)^{\frac{5}{2}}$.

39. $\left(\dfrac{35}{113}\right)^{\frac{3}{8}}$.

40. $\left(\dfrac{.08726}{.1321}\right)^{\frac{5}{3}}$.

41. $\sqrt[8]{\dfrac{21}{13}}$.

42. $\sqrt[9]{-\dfrac{3}{7}}$.

43. $\sqrt[5]{\dfrac{2}{3}} \div \sqrt[3]{\dfrac{3}{5}}$.

44. $\sqrt[8]{2} \times \sqrt[5]{3} \times \sqrt[7]{.01}$.

45. $\sqrt[5]{\dfrac{3258}{49309}}$.

46. $\left(\dfrac{-31.63}{429}\right)^{\frac{3}{17}}$.

47. $\dfrac{100^{\frac{2}{3}}}{(.7325)^{\frac{3}{7}}}$.

48. $\dfrac{\sqrt[3]{.0001289}}{\sqrt[4]{.0008276}}$.

49. $\dfrac{(-.7469)^{\frac{5}{3}}}{-(.2345)^{\frac{7}{2}}}$.

50. $\dfrac{\sqrt[11]{.0073}}{(.68291)^{\frac{5}{2}}}$.

51. $\dfrac{\sqrt{5.955} \times \sqrt[3]{61.2}}{\sqrt[5]{298.54}}$.

52. $(538.2 \times .0005969)^{\frac{1}{8}}$.

53. $(18.9503)^{11} \times (-.1)^{14}$.

54. $\sqrt[6]{3734.9 \times .00001108}$.

55. $(2.6317)^{\frac{3}{4}} \times (.71272)^{\frac{2}{5}}$.

56. $\dfrac{\sqrt[3]{-.008193} \times (.06285)^{\frac{3}{2}}}{-.98342}$.

57. $\sqrt{.035} \times \sqrt[6]{.62667} \times \sqrt[3]{.0072103}$.

EXPONENTIAL EQUATIONS.

362. An **Exponential Equation** is one in which the unknown quantity occurs as an exponent.

To solve an equation of this form, take the logarithms of both members; the result will be an equation which can be solved by ordinary algebraic methods.

363. **1.** Given $31^x = 23$; find the value of x.

Taking the logarithms of both members,

$$\log(31^x) = \log 23.$$

Or (Art. 346),

$$x \log 31 = \log 23.$$

Whence, $$x = \frac{\log 23}{\log 31} = \frac{1.3617}{1.4914} = .91303, \textit{ Ans.}$$

2. Given $.2^x = 3$; find the value of x.

Taking the logarithms of both members,

$$x \log .2 = \log 3.$$

Whence, $$x = \frac{\log 3}{\log .2} = \frac{.4771}{9.3010-10} = \frac{.4771}{-.6990}$$
$$= -.6825, \textit{ Ans.}$$

EXAMPLES.

Solve the following equations:

3. $11^x = 3.$ **5.** $13^x = .281.$ **7.** $a^x = b^m c^n.$

4. $.3^x = .8.$ **6.** $.703^x = 1.096.$ **8.** $ma^{\frac{1}{x}} = n.$

9. Given a, r, and l; derive the formula for n. (Art. 322.)

10. Given a, r, and S; derive the formula for n.

11. Given a, l, and S; derive the formula for n.

12. Given r, l, and S; derive the formula for n.